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Category Theory and Homological Algebra

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Part I

Category Theory

Chapter 1

Basic concepts of category theory

Ah, that will never prove it.
— Neru, *Abstract Nonsense*

1.1 Metacategory[1]

Definition 1.1.1. A **metagraph** consists of **objects** and **arrows**, with two operations taking each arrow to object:

- **Domain**, which assigns to each arrow f an object $\operatorname{dom} f$;
- **Codomain**, which assigns to each arrow f an object $\operatorname{cod} f$.

If so, we write $f : \operatorname{dom} f \to \operatorname{cod} f$, or $\operatorname{dom} f \xrightarrow{f} \operatorname{cod} f$. [2]

The visualization of a metagraph, by using the (possibly labelled) objects and arrows, is called the **diagram** of the metagraph. We call the diagram **commutes** if any two (directed) routes connecting two objects are equivalent[3].

Definition 1.1.2. A **metacategory** is a metagraph with two operations:

- **Identity**, which assigns to each object a an arrow $1_a : a \to a$[4];
- **Composition**, which assigns to each pair $\langle g, f\rangle$ of arrows with $\operatorname{dom} g = \operatorname{cod} f$ an arrow $g \circ f : \operatorname{dom} f \to \operatorname{cod} g$.[5]

These operations satisfies the two following axioms:

- **Unit law**: for all arrows $f : a \to b$, $1_b \circ f = f = f \circ 1_a$.
- **Association law**: for all arrows $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, the following equality holds.[6]

$$k \circ (g \circ f) = (k \circ g) \circ f \tag{1.1}$$

We often write $g \circ f$ as gf.

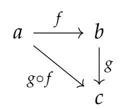
[1] In this subsection, we ignore all the set-theoretical problems, by throwing some axioms in a vacant logical space.

[2] This indeed is a very simple example of the diagram, defined below, consists of two objects $\operatorname{dom} f, \operatorname{cod} f$ and one arrow f.

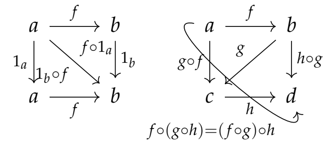
[3] The equivalence relation on arrows depends on the context, for example, the associativity defined in metacategory below.

[4] Because every identity arrow corresponds to every object, we can define the category by only using morphisms.

[5] The diagrammatic description of composition is the following.



[6] The diagrammatic description of unit law(left) and composition(right) is the following.



| Example 1.1.3. |

1. The **metacategory of sets** is a metcategory whose objects are all sets and arrows are all functions, with usual identity function and composition between two functions.
2. The **metacategory of groups** is a metacategory wose objects are all groups and arrows are all homomorphisms, with usual identity morphism and composition between two morphisms.
3. There are also many other metacategories: **rings** with ring homomorphisms, **fields** with field homomorphisms, **topological spaces** with continuous maps, etc.

1.2 Category

| Definition 1.2.1. | A **universe** is a set U with the following properties:[7]

1. $\emptyset \in U$ (empty set rule);
2. $x \in u \in U$ implies $x \in U$ (transitive rule);
3. $u \in U$ implies $\{u\} \in U$ (singleton set rule);
4. $x \in U$ implies $\mathcal{P}(x) \in U$ (power set rule);
5. For $I \in U$ and $\{x_\alpha\}_{\alpha \in I}$, $\cup_{\alpha \in I} x_\alpha \in U$ (union rule);
6. $\omega \in U$, where ω is the set of all finite ordinals (infinite set rule).

We call a set u a U**-set** if $u \in U$, and a set u a U**-small set** if u is isomorphic to a U-set.[8]

[7] Because of the proper class problem, or Russel's paradox, we need to restrict down the number of targets setwisely. First four statements shows that all the Zermelo-Fraenkel(ZF) axiomatic operations works in U, the fifth statement allows the structure of well-known arithmetic, and the last statement allows the structure of well-known functions.

[8] If the universe U is already given, then we simply say U-set a set and U-small set a small set.

| Proposition 1.2.2. | *Let U be a universe.*

1. *$u \in U$ implies $\cup_{x \in u} x \in U$.*
2. *$u \subset v \in U$ implies $u \in U$.*
3. *$u, v \in U$ implies $u \times v \in U$.*
4. *$I \in U$ and $u_i \in U$ for all $i \in I$ implies $\prod_{i \in I} u_i \in U$.*

Proof. 1. From the union rule, take I as u, and we take $\{x_\alpha\}_{\alpha \in I}$ as u itself. Then $\cup_{x \in u} x \in U$.
2. $u \subset v$ implies $u \in \mathcal{P}(v)$ and because $v \in U$, by power rule, $\mathcal{P}(v) \in U$, and by transition rule, $u \in U$.
3. Because $u \times v \in \mathcal{P}(\mathcal{P}(\mathcal{P}(u \cup v)))$, $u \times v \in U$.
4. Because $\prod_{i \in I} u_i \in \mathcal{P}(\mathcal{P}(I \times \cup_{i \in I} u_i))$, $\prod_{i \in I} u_i \in U$.

□

| Definition 1.2.3. | A **graph** is a set of objects and a set of arrows, with two functions dom, cod from morphisms to objects. We call the arrows in category as **morphisms**.

A **category** C is a graph which is also a metacategory, that is, it has two additional functions, identity and composition, all of those satisfies the condition of metacategory.[9] We write ob C as the set of

[9] Thus we may consider the category as the metacategory which can be treated under set theory, or more explicitly, a universe.

objects in C, $\operatorname{mor}\mathsf{C}$ as the set of **morphisms in** C, and $\hom_{\mathsf{C}}(a,b)$ as the set of **morphisms in** C **with domain** a **and codomain** b.

We frequently write $a \in \operatorname{ob}\mathsf{C}$ as $a \in \mathsf{C}$ and $f \in \hom_{\mathsf{C}}(a,b)$ as $f \in \hom(a,b)$, $f \in \mathsf{C}(a,b)$ or $f \in \mathsf{C}$, when all the ignored elements are assumed to be known in the context.

| Definition 1.2.4. | Fix a universe U.[10]

If C is a category with small $\operatorname{ob}\mathsf{C}$ and $\hom(a,b) \in \mathsf{C}$ are small for all $a, b \in \mathsf{C}$, then we call it a **locally small category**.

We call C a **small category** if C is locally small category[11] and $\operatorname{mor}\mathsf{C}$ is small.

[10] From now, we will not mention what universe we are working on.

[11] Indeed, we only need small $\operatorname{ob}\mathsf{C}$ rather then local smallness, because the smallness of $\operatorname{mor}\mathsf{C}$ gives the smallness of all $\hom(a,b)$.

| Example 1.2.5. |

1. Set is the category whose objects are sets and morphisms are functions. This is same for Group with group homomorphisms, Meas with measurable functions, Top with continuous functions, Man with continuous functions on manifold, and Poset with order-preserving functions on Partially ordered set, and so on.
2. Consider a group G. Then the category $\mathsf{B}G$ is an one-object category defined by G, where the morphisms are the elements of G. Here, the composition of morphisms are defined by the multiplication of group elements.
3. Consider a poset P. Then the category P is a category with its elements as objects and $f : x \to y$ as morphisms for all $x \le y$.
4. The category $\mathsf{0}$ is a category with no object and no morphism.
5. The category $\mathsf{1}$ is a category with one object and one morphism, the identity.
6. The category $\mathsf{2}$ is a category with two objects, two identities, and one morphism between them.

2020.12.12.

1.3 Morphism

| Definition 1.3.1. | Consider a morphism $f \in \mathsf{C}(X,Y)$. Then f is called an **isomorphism** if $fg = 1_Y$ and $gf = 1_X$, and call f an **inverse** of g. If so, we say X, Y are **isomorphic**, and write $X \simeq Y$.

If a morphism $f \in \mathsf{C}$ satisfies $\operatorname{dom} f = \operatorname{cod} f$, then we call it **endomorphism**.

If an endomorphism is also an isomorphism, then we call it **automorphism**.

| Example 1.3.2. |

1. For the category Set, the isomorphisms are bijections. Similarly, we have group isomorphisms for Group, measurable bijections for Meas, homeomorphisms for Top and Man, and order isomorphisms for Pos.
2. For a group G, every morphisms in $\mathsf{B}G$ are automorphisms.

3. For a poset P, only identities are isomorphisms, hence automorphisms.

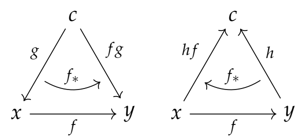
| **Lemma 1.3.3.** | *For each morphism $f \in \mathsf{C}$, there is at most one inverse of f.*

Proof. Suppose g, h are inverses of f. Then $gfh = (gf)h = h$, but also $gfh = g(fh) = g$, hence $g = h$. □

| **Definition 1.3.4.** | Let C be a locally small category, and $f : x \to y \in \mathsf{C}$ be a morphism. Choose an object c.[12]

1. The **post-composition** $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is a function taking $g : c \to x$ to $f_*(g) = fg : c \to y$.
2. The **pre-composition** $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is a function taking $h : y \to c$ to $hf : x \to c$.

[12] The following two diagrams show the post(left) and pre(right) composition.



| **Lemma 1.3.5.** | *Let $f : x \to y \in \mathsf{C}$[13]. Then the followings are equivalent.*

1. *f is an isomorphism in C.*
2. *For any object $c \in \mathsf{C}$, the post-composition $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is a bijection.*
3. *For any object $c \in \mathsf{C}$, the pre-composition $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is a bijection.*

[13] From now, we consider a category C as a locally small category, unless mentioned.

Proof.

(1 ⇒ 2). Because f is an isomorphism, we have its inverse $g : y \to x$. Thus for all $h \in \mathsf{C}(c, x)$,

$$g_* f_*(h) = gfh = h \tag{1.2}$$

hence $g_* f_* = 1_{\mathsf{C}(c,x)}$ Similarly, for all $k \in \mathsf{C}(c, y)$,

$$f_* g_*(k) = fgk = k \tag{1.3}$$

hence $f_* g_* = 1_{\mathsf{C}(c,y)}$.

(2 ⇒ 1). Because f_* is a bijection, we have $g := f_*^{-1}(1_y)$. By definition $fg = 1_y$. Now because $f_*(gf) = fgf = f = f_*(1_x)$ and f_* is bijective, $gf = 1_x$.

(1 ⇔ 3). This case can be proven by almost similar[14] way above.

□

[14] Here, the almost similarness, which has a proof with only opposite arrows, is called the **dual theorem**. This will be discussed in the Section 1.7.

| **Definition 1.3.6.** | Let $f : x \to y \in \mathsf{C}$ be a morphism.

1. We call f a **monomorphism**, or in short **monic**, if for any morphisms $h, k : w \to x$, $fh = fk$ implies $h = k$.
   To say f is monic, we write $f : x \rightarrowtail y$.
2. We call f an **epimorphism**, or in short **epi**, if for any morphisms $p, q : y \to z$, $pf = qf$ implies $p = q$.
   To say f is epi, we write $f : x \twoheadrightarrow y$.

Proposition 1.3.7. *Consider a morphism $f \in \mathsf{C}$.*[15]

1. *f is monic if and only if $f_* : \mathsf{C}(c,x) \to (c,y)$ is injective for all $c \in \mathsf{C}$.*
2. *f is epi if and only if $f^* : \mathsf{C}(y,c) \to \mathsf{C}(x,c)$ is injective for all $c \in \mathsf{C}$.*

[15] We will consider the surjective pre- and post-composition cases in Theorem 1.3.12.

Proof.

1.

($\Rightarrow$). For any $c \in \mathsf{C}$ and $h,k \in \mathsf{C}(c,x)$, $fh = fk$ implies $h = k$. Now $f_*(h) = f_*(k)$ is equivalent with $fh = fk$.

($\Leftarrow$). Choose $c \in \mathsf{C}$. Because $f_* : \mathsf{C}(c,x) \to \mathsf{C}(c,y)$ is injective, for all $h,k \in \mathsf{C}(c,x)$, $f_*(h) = f_*(k)$ implies $h = k$. Now $fh = fk$ is equivalent with $f_*(h) = f_*(k)$.

2. The proof can be done by almost similar argument with above.[16]

□

[16] Again, the almost similarness implies the **dual theorem**. From now we will skip all the dual theorem proofs.

Proposition 1.3.8. *Consider the function $f \in \mathsf{Set}(X,Y)$.*

1. *f is monic if and only if f is injective.*
2. *f is epi if and only if f is surjecetive.*[17]

[17] This is **NOT** the dual statement, because the space $\mathsf{Set}^{\mathrm{op}}$ does not have same structure with Set. If we consider the statement *f is surjective if and only if f has its right inverse*, then its dual statement says *f is injective if and only if f has its left inverse*, so we can use the duality property. However because the first statement is equivalent with Axiom of Choice, it is an overkill.

Proof.

1. For any $x \in X$, define $1_x : \{\bullet\} \to X$ with $1_x(\bullet) = x$.[18] Then $f_*(1_x) = f_*(1_{x'})$ if and only if $f(x) = f(x')$. Because $1_x = 1_{x'}$ if and only if $x = x'$, f_* is injective if and only if f is injective. By the proposition 1.3.7, the given statement holds.
2.

($\Rightarrow$) Take $y \in Y - f(X)$ and define $h : Y \to \{0,1\}$ as $h(Y) = 0$ and $k : Y \to \{0,1\}$ as $k^{-1}(1) = \{y\}$. Then $hf = kf$ but $h \neq k$, contradiction, thus $Y = f(X)$.

($\Leftarrow$) Consider $h,k : Y \to Z$ with $hf = kf$. Because f is surjective, $k(y) = kf(f^{-1}(y)) = hf(f^{-1}(y)) = h(y)$ for all $y \in Y$. Hence $k = h$.

□

[18] Notice that we can consider element because X is a set. In general, even if a category C has exactly same structure with Set, we may cannot choose an element in any object of C, because it is not necessary to define the category. However we may choose a 'one-set like' object $\bullet$ in C, and consider $\mathsf{C}(\bullet, X)$ as the set of X, defining elements categorically. We will later discuss when we can find such 'one-set like' object, or also called as, the **terminal object**. The dual concept of it is the **initial object**.

Example 1.3.9. [19]

Consider the inclusion mapping $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring. Then i is monic and epi, but not isomorphic.

Indeed, for $h,k : R \to \mathbb{Z}$, $ih = ik$ implies $ih(r) = ik(r)$ thus $h(r) = k(r)$ for all $r \in R$, hence $h = k$ and i is Monic.

Also, for $h,k : \mathbb{Q} \to R$, if $hi = ki$ but $h \neq k$ then we have $q \in \mathbb{Q}$ such that $h(q) \neq k(q)$. Because $q \notin \mathbb{Z}$, $q = r/p$ for some relatively prime integers r,p. Then $h(r) = k(r)$ thus $p \cdot h(q) \neq p \cdot k(q)$, contradiction.

Finally, consider the nontrivial ring homomorphism $f : \mathbb{Q} \to \mathbb{Z}$. Then $f(q) = n \neq 0$ for some $q \in \mathbb{Q}$, with $n = 2^a m$ with odd m. Then $f(q/2^{a+1}) = m/2 \notin \mathbb{N}$, contradiction. Hence i is not an isomorphism.

[19] This example shows that monic and epi does not implies isomorphic.

Definition 1.3.10. Let $s \in \mathsf{C}(x,y)$ and $r \in \mathsf{C}(y,z)$ such that $rs = 1_x$.

1. We call s a **section**, **split monomorphism**, or **right inverse** of r.
2. We call r a **retraction**, **split epimorphism**, or **left inverse** of s.[20]

[20] The concepts, split monomorphism and split epimorphism, are dual to each other.

Proposition 1.3.11. *Let $f \in \mathsf{C}(x,y)$.*

1. *If f is a split monic, then f is monic.*
2. *If f is a split epi, then f is epi.*

Proof.

1. We have $g \in \mathsf{C}(y,x)$ such that $gf = 1_x$. Now if $fh = fk$ for some $h,k \in \mathsf{C}(w,x)$, then $h = gfh = gfk = k$.
2. Similar as above.

□

Theorem 1.3.12. [21] *Let $f \in \mathsf{C}(x,y)$.*

1. *f is a split epimorphism if and only if $f_* : \mathsf{C}(c,x) \to \mathsf{C}(c,y)$ is surjective.*
2. *f is a split monomorphism if and only if $f^* : \mathsf{C}(y,c) \to \mathsf{C}(x,c)$ is surjective.*

[21] Compare this with Proposition 1.3.7.

Proof.

1.

($\Rightarrow$). Because f is a split epimorphism, we have $g \in \mathsf{C}(y,x)$ such that $fg = 1_y$. Now consider $k \in \mathsf{C}(c,y)$. Then $gh \in \mathsf{C}(c,x)$ satisfies $f_*(gh) = h$, thus f_* is surjective.

($\Leftarrow$). We have $g \in f_*^{-1}(1_y)$, which gives $fg = 1_y$.

2. Similar as above.

□

Corollary 1.3.13. *Let $f \in \mathsf{C}(x,y)$ be a morphism. Then the followings are equivalent.*

1. *f is isomorphic.*
2. *f is monic and split epi.*
3. *f is epi and split monic.*

Proof.

(1 $\Leftrightarrow$ 2). By Proposition 1.3.7, f is monic if and only if f_* is injective. By Theorem 1.3.12, f is split epi if and only if f^* is surjective. By Lemma 1.3.5, f is isomorphic if and only if f^* is bijective. Because f^* is bijective if and only if f^* is injective and surjective, the statement is true.

(1 $\Leftrightarrow$ 3). Similar as above.

□

Lemma 1.3.14. *Let $f \in \mathsf{C}(x, y)$ and $g \in \mathsf{C}(y, z)$.*[22]

1. *If $f : x \rightarrowtail y$ and $g : y \rightarrowtail z$ are monic, then $gf : x \rightarrowtail z$ is monic.*
2. *If $f : x \to y$ and $g : y \to z$ gives monic composition $gf : x \rightarrowtail z$, then f is monic.*

1′. *If $f : x \twoheadrightarrow y$ and $g : y \twoheadrightarrow z$ are epi, then $gf : x \twoheadrightarrow z$ is epi.*

2′. *If $f : x \to y$ and $g : y \to z$ gives epi composition $gf : x \twoheadrightarrow z$, then g is epi.*

[22] The number of statement shows the dual relation: $(1 \leftrightarrow 1')$ and $(2 \leftrightarrow 2')$.

Proof.

1. For $h, k \in \mathsf{C}(w, x)$, because g is monic, $gfh = gfk$ implies $fh = fk$, and because f is monic, $h = k$.
2. For $h, k \in \mathsf{C}(w, x)$, suppose that $fh = fk$. Then $gfh = gfk$ thus $h = k$.

1′. Similar as 1.

2′. Similar as 2.

□

2020.12.14.

1.4 *Functor*

Definition 1.4.1. Let C, D be categories. A **functor** $F : \mathsf{C} \to \mathsf{D}$ consists of the following data:[23]

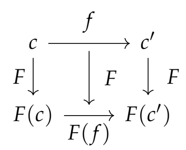
- An object $F(c) \in \mathsf{D}$ for each object $c \in \mathsf{C}$;
- A morphism $F(f) : F(c) \to F(c') \in \mathsf{D}$ for each morphism $f : c \to c' \in \mathsf{C}$.

These data satisfies the following **functoriality axioms**:[24]

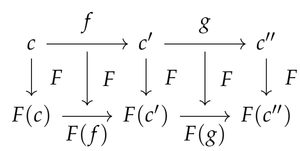
- For any composable morphism pair $f, g \in \mathsf{C}$, $F(g)F(f) = F(gf)$;
- For each object $c \in \mathsf{C}$, $F(1_c) = 1_{F(c)}$.

For any two functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$, we have a **composite functor** $G \circ F : \mathsf{C} \to \mathsf{E}$, also written as GF, defined as $GF(c) = G(F(c))$ and $GF(f) = G(F(f))$.

[23] We can draw the data of functor as following.



[24] The composition rule can be drawn as following.



Example 1.4.2.

1. The **forgetful functor** is the functor $F : \mathsf{C} \to \mathsf{D}$, which "forgets" some property of category C[25].
   For example, let C be one of the categories
   $\mathsf{Group}, \mathsf{Ring}, \mathsf{Mod}_R, \mathsf{Field}, \mathsf{Meas}, \mathsf{Top}, \mathsf{Poset}$, or any other set-based category. The forgetful functor F takes $c \in \text{ob}\mathsf{C}$ to the set c, and takes $f \in \text{mor}\mathsf{C}$ to the function f. In each cases, we are forgetting certain properties which characterize the category.
   Considering Set as the base category, we have seen the "fully" forgetful functors. We can also define the "partial" forgetful functors. For example:

[25] The concept "forgetful functor" does not have any precise definition. Indeed, mostly we use this terminology from a set-like category to Set, which is explained below.

(a) The functor $F : \mathsf{Ring} \to \mathsf{CRing}$ from ring category to commutative ring category forgetting commutator;
(b) The functor $F : \mathsf{Mod}_R \to \mathsf{Ab}$ from R-module category to abelian group category forgetting Modular properties;
(c) The functor $\mathsf{CRing} \to \mathsf{Ab}$ from commutative ring category to abelian group category forgetting multiplicative properties;

and so on.

2. In topology, consider a functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Group}$ from point-fixed topological set category to group category, taking (X, x) to its fundamental group $\pi_1(X, x)$ and $f : (X, x) \to \pi_1(Y, y)$ to the induced homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$.
3. Consider the functors $Z_n, B_n, H_n : \mathsf{Ch}_R \to \mathsf{Mod}_R$ from R-chain complex category to R-module category. Here the ***n*-cycle** is defined as $Z_n(C_\bullet) = \ker(d : C_n \to C_{n-1})$, the ***n*-boundary** as $B_n(C_\bullet) = \mathrm{Im}(d : C_{n+1} \to C_n)$, and the ***n*-th homology** as $H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet)$.
4. Consider the functor $F : \mathsf{Set} \to \mathsf{Group}$. Here $F(X)$ is the free group generated by the set X. This functor, indeed, satisfies some **universal property**, which we will discuss later.

| Example 1.4.3. | [26]

1. Consider the functor $\star : \mathsf{Vect}_k \to \mathsf{Vect}_k^{\mathrm{op}}$, taking a vector space V to its **dual space** $V^* = \hom(V, k)$. Then the linear map $\phi : V \to W$ gives the arrow $\phi^\star : V^* \to W^*$, which exists when we have a dual map $\phi^* : W^* \to V^*$ with $\phi^*(f : W \to k) = f \circ \phi$.[27]
2. Consider the functor $\mathrm{Spec} : \mathsf{CRing}^{\mathrm{op}} \to \mathsf{Top}$, taking a commutative ring R to the set of prime ideals $\mathrm{Spec}R$ with Zariski topology. Then the ring homomorphism $\phi : R \to S$ gives the arrow $\mathrm{Spec}\phi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, which exists when we have an inverse map $\phi^{-1} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$.
3. Consider the functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ for some arbitrary category C[28] For example, consider a category $\mathcal{O}(X)$ for some topological space X, which is the poset category of open subsets of X. If $V \subset U$, we have $\mathrm{res}_{V,U} : F(U) \to F(V)$. We call such functor F a **presheaf**.
4. Consider the functor $F : \mathcal{O}(X) \to \mathsf{Ring}$, defined as $F(X)$ be the bounded functions on U and $F(V \subset U)$ be the restriction function $\mathrm{res}_{V,U} : F(U) \to F(V)$ taking f to $f|_V$. By taking forgetful functor, we can consider F as a presheaf. If, furthermore, suppose that for all $V, W \subset U$ and $g \in F(V), h \in F(W)$ satisfying $g|_{V\cap W} = h|_{V\cap W}$, we have $f \in F(U)$ such that $f|_V = g$ and $f|_W = h$. Then we call F a **sheaf**.

| Lemma 1.4.4. | *Functors preserve split monics, split epis, and isomorphisms.*

[26] The reason why we write these functors separately is, indeed, we may consider these functors as a functor from an **opposite category** to a category. The opposite category, closely related with the duality, will be treated later.

[27] Here, to us, it will be easier to define the directions of arrow oppositely in $\mathsf{Vect}_k^{\mathrm{op}}$. This direction change under $\star$ indeed shows the 'contravarient property' of $\star$, so will be called the **contravariant functor**. On the other hand, the functors above are defined covariently, thus they are called the **covariant functor**.

[28] Here we can see that we are considering op as the functor acting on C, giving reversed direction of arrow. We will explicitly define this concept in Section 1.7.

Proof. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor and $f \in \mathsf{C}(x, y)$ be split monic. Then we have

$$F(g)F(f) = 1_{F(x)} \tag{1.4}$$

thus $F(f)$ is a split monic.

Split epi case can be done similarly.

From Corollary 1.3.13, f is isomorphic if and only if f is split monic and split epi. Thus functor preserve isomorphisms. □

| EXAMPLE 1.4.5. | [29] Consider a category 2, consists of two objects and one arrow $\bullet \to \circ$(except identities). Then the arrow is monic and epi.

Consider a functor $F : (\bullet \to \circ) \to \mathsf{Mod}_{\mathbb{Z}}$, defined as $F(\bullet) = F(\circ) = \mathbb{Z}$ and $F(\bullet \to \circ) : \mathbb{Z} \to \mathbb{Z}$ becomes a trivial map $n \mapsto 0$. Then $F(\bullet \to \circ)$ is neither monic nor epi.

[29] This example shows that the functors need not preserve monics and epis.

| EXAMPLE 1.4.6. | [30] Let C be a category with two objects and one arrow(except identity), $\bullet \to \circ$. Let C be a category with two objects and two arrows(except identity), $\bullet \leftrightarrows \circ$.

Let $F : \mathsf{C} \to \mathsf{D}$ be a functor taking objects and arrows to itself. Then $F(\bullet \to \circ)$ is an isomorphism, but $\bullet \to \circ$ is not.

[30] This example shows that the functors need not reflect isomorphisms.

| DEFINITION 1.4.7. | A functor $F : \mathsf{C} \to \mathsf{D}$ is **conservative** if it reflects isomorphisms. That is, For all morphisms $f \in \mathsf{C}(x, y)$, if $F(f)$ is isomorphic, then f is isomorphic.

| DEFINITION 1.4.8. | A **subcategory** of C is a collection of some of the objects and some of the arrows of C, which is itself a category.

Let D be a subcategory of C. The inclusion map $\mathsf{D} \to \mathsf{C}$, taking each object and each arrow in D to itself in C, is called the **inclusion functor**.

| DEFINITION 1.4.9. | Let $F : \mathsf{C} \to \mathsf{D}$ be a functor.

- We say F is a **full** if for each objects $x, y \in \mathsf{C}$, $F : \mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is surjective.
- We say F is a **faithful** if for each objects $x, y \in \mathsf{C}$, $F : \mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is injective.
- We say F is **essentially surjective on objects** if for every object $d \in \mathsf{D}$, there is an object $c \in \mathsf{C}$ such that $F(c) \simeq d$.[31]
- We say F is an **embedding** if it is faithful functor and $F : \mathrm{ob}\,\mathsf{C} \to \mathrm{ob}\,\mathsf{D}$ is injective.
- We say F is **fully faithful** if it is full and faithful.
- We say F is **full embedding of** C **into** D if it is full and embedding. If so, then we say C is a **Full subcategory** of D.

[31] Notice that they do not need to be exactly same.

| PROPOSITION 1.4.10. | *The image of full embedding functor $F : \mathsf{C} \to \mathsf{D}$ is a subcategory of* D.[32]

[32] This also holds on the fully faithful functor, but not exactly. Indeed, any fully faithful functor is **equivalent** to some full embedding functor. We will discuss this after when we discuss the equivalence between functors.

Proof. We only need to check that $F(C)$ is indeed a category. The associativity, injectivity of objects and morphisms, and existence of identity holds naturally. For the composition rule, because $\mathsf{C}(x,y) \simeq \mathsf{C}(F(x),F(y))$ for all objects $x,y \in \mathsf{C}$, there always exists $F(g)F(f) = F(gf)$ for all composable f,g. $\square$

Example 1.4.11.

1. The forgetful functor $\mathsf{Group} \to \mathsf{Set}$ is faithful, but not full and essentially surjective on objects.
2. Consider the functor from $\mathsf{B}\mathbb{Z}/4 \to \mathsf{B}\mathbb{Z}/2$, which is the nontrivial homomorphism. This functor is full and essentially surjective on objects, but not faithful.
3. Consider the category C with four objects $\{a,b,c,d\}$ and two nontrivial morphisms $a \to b, c \to d$. Also consider the category D with three objects $\{x,y,z\}$ and three nontrivial morphisms $x \to y \to z, x \to z$.
   Define a functor F as following. On objects, $F(a) = x, F(b) = F(c) = y, F(d) = z$. All morphisms are defined accordingly.
   Then F is embedding, but not full. Indeed, image has $x \to y \to z$, but does not have their composition $x \to z$. Thus the image of F does not give a subcategory of D.

Definition 1.4.12. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors. If $FG = 1_{\mathsf{D}}$ and $GF = 1_{\mathsf{C}}$, then we call F, G as the **isomorphisms of categories**, and we say C, D are **isomorphic categories**.

Example 1.4.13.

1. For a group G, the functor $-1 : \mathsf{B}G \to \mathsf{B}G^{\mathrm{op}}$, taking $g \to g^{-1}$, is isomorphic.
2. Let E/F be a finite Galois extension and $G := \mathrm{Aut}(E/F)$ the Galois group.
   Define **orbit category** $\mathcal{O}_G$, whose objects are cosets G/H and morphisms $f : G/H \to G/K$ are G**-equivariant maps**, satisfying $g' f(gH) = f(g'gH)$. Indeed every G-equivariant map can be represented as $gH \mapsto g\gamma K$, for some $\gamma \in G$ with $\gamma^{-1}H\gamma \subset K$.
   Define the category Field_F^E, whose objects are intermediate fields $E/K/F$, and morphisms $f : K \to L$ are the field homomorphisms fixing F.
   Now we define $\Phi : \mathcal{O}_G^{\mathrm{op}} \to \mathsf{Field}_F^E$, taking objects G/H to the H-fixed subfield, and morphisms $G/H \to G/K$ induced by γ to the field homomorphism $x \mapsto \gamma x$ from K-fixed subfield to H-fixed subfield.
   The **fundamental theorem of Galois theory** then says Φ is isomorphic.

Example 1.4.14. [33] Consider a category Set^{∂}, whose objects are sets and morphisms are **partial functions**: $f : X \to Y$ is a function from $X' \subset X$ to Y.

Consider the category Set_*, whose objects are **pointed sets** (X, x), the sets X with a freely-added basepoint $x \in X$, and morphisms are the functions.

Take the functor $(-)_+ : \mathsf{Set}^{\partial} \to \mathsf{Set}_*$, which sends X to the pointed set $X_+ := (X \cup \{X\}, \{X\})$, and the partial function $X \to Y$ to the pointed function $f_+ : X_+ \to Y_+$, where all the elements out of the domain of function f maps to the basepoint of Y_+.

Take the functor $U : \mathsf{Set}_* \to \mathsf{Set}^{\partial}$, which sends X_+ to the set X, and the pointed function $f_+ : X_+ \to Y_+$ to the partial function f on X.

Because of the construction, $U(-)_+ = 1_{\mathsf{Set}^{\partial}}$. However, $(U-)_+$ sends (X, x) to $(X - \{x\} \cup \{X - \{x\}\}, X - \{x\})$. This is isomorphic but not identical, hence $(U-)_+ \neq 1_{\mathsf{Set}_*}$, and so U and $(-)_+$ are not the isomorphisms.

[33] This example shows that the isomorphisms of categories are not very useful if we want to compare the structure between two categories. Hence we use the concept **natural transformation**, which will be discussed in the Section 1.5.

1.5 *Natural Transformations*

Definition 1.5.1. Consider functors $F, G : \mathsf{C} \to \mathsf{D}$. Then a **natural transformation** $\alpha : F \Rightarrow G$ consists of following data:

- For every objects $c \in \mathsf{C}$, we have a morphism $\alpha_c : F(c) \to G(c)$ in D.

These morphisms must satisfy the following statement:

- For any morphism $f : c \to c'$ in C, $G(f)\alpha_c = \alpha_{c'}F(f)$.[34]

[34] This relation can be drawn as following.

$$\begin{array}{ccc} F(c) & \xrightarrow{\alpha_c} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow[\alpha_{c'}]{} & G(c') \end{array}$$

Example 1.5.2.

1. Consider the vector space V over the field k. Then the map $\mathrm{ev}_V : V \to V^{**}$, taking $v \in V$ to $\mathrm{ev}_V(v) : V^* \to k$, are the components of a natural transformation from 1_{Vect_k} to the double dual functor $**$.[35]
2. Consider the finite vector space V over the field k. Then the identity functor and dual $*$-functor are not natural transformations, because the identity functor does not changes the direction of arrow, but the dual functor does.
3. Consider a category of commutative ring cRing and a category of group Group. From a commutative ring R, we may consider the general linear group $GL_n(R)$ and the group of units $R^\times$. Thus, GL_n and $(-)^\times$ are functors from cRing to Group.
   Now consider the determinant $\det_R : GL_n(R) \to R^\times$. Then for any ring homomorphism $\phi : R \to S$, we have $\det_S(GL_n(\phi)) = \phi^\times \circ \det_R$,[36] thus $\det : GL_n \Rightarrow (-)^\times$ is the natural transformation.

[35] This relation can be drawn as following.

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{ev}_V} & V^{**} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi^{**}} \\ W & \xrightarrow{\mathrm{ev}_W} & W^{**} \end{array}$$

[36] This relation can be drawn as following.

$$\begin{array}{ccc} GL_n(R) & \xrightarrow{\det_R} & R^\times \\ {\scriptstyle GL_n(\phi)}\downarrow & & \downarrow{\scriptstyle \phi^\times} \\ GL_n(S) & \xrightarrow{\det_S} & S^\times \end{array}$$

1.6

2020.12.21.

Equivalence on Categories

Definition 1.6.1. Let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be the functors. We call F, G an **equivalence of categories** if there is a natural isomorphisms $\eta : 1_{\mathsf{C}} \simeq G \circ F$ and $\epsilon : F \circ G \simeq 1_{\mathsf{D}}$, and G a **quasi-inverse** of F and vice versa. If so, we call categories C and D are **equivalent**, and write $\mathsf{C} \simeq \mathsf{D}$.

Proposition 1.6.2. *The equivalence of categories is an equivalence relation.*

Proof. Suppose that $\mathsf{C} \simeq \mathsf{D} \simeq \mathsf{E}$ with $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ and $H : \mathsf{D} \leftrightarrows \mathsf{E} : K$, which are equivalence of categories. Then $H \circ F : \mathsf{C} \leftrightarrows \mathsf{E} : G \circ K$ are equivalence of categories. □

Theorem 1.6.3 (Characterizing equivalences of categories).

1. *An equivalence of categories functor is fully faithful and essentially surjective on objects.*
2. *Assuming the axiom of choice, any fully faithful functor which is essentially surjective on objects defines an equivalence of categories.*

Proof.

1. Let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ such that $\eta : 1_{\mathsf{C}} \simeq GF$ and $\epsilon : 1_{\mathsf{D}} \simeq FG$.[37]
   For all objects $d \in \mathsf{D}$ we have $FG(d) \simeq d$, hence F is essentially surjective on objects. Same holds for G.
   For two $f, g : c \to c' \in \mathsf{C}$, if $F(f) = F(g)$ then $GF(f) = GF(g)$, implying $f = g$.[38] Hence F is faithful. Same holds for G.
   Finally, for all morphism $k : F(c) \to F(c')$, $G(k) : GF(c) \to GF(c')$. Then because η is isomorphic, $\eta_{c'} h = G(k)\eta_c$, and because $\eta_{c'} \circ h = GF(h) \circ \eta_c$ by definition, we get $G(k) = GF(h)$. Because G is faithful, $k = F(h)$. Hence F is full. Same holds for G.
2. Suppose that $F : \mathsf{C} \to \mathsf{D}$ is a fully faithful functor which is essentially surjective on objects. Because of the essential surjectivity on objects, for each object $d \in \mathsf{D}$ there is a nonempty subcollection of the objects C which becomes d under F. Using the axiom of choice[39], we can choose $G(d) \in \mathsf{C}$ such that we have $\epsilon_d : F(G(d)) \simeq d$. Also because F is fully faithful, for every morphism $f : d \to d' \in \mathsf{D}$, there is a morphism $G(f) : G(d) \to G(d')$ which satisfies $f \circ \epsilon_d = \epsilon_{d'} \circ F(G(f))$.[40] Hence, if G is a functor, then $\epsilon : FG \Rightarrow 1_{\mathsf{D}}$ is a natural transformation.
   To show that G is actually functor, we need to show that G conserves identity morphism and morphism composition. For the identity, choose an object $d \in \mathsf{D}$. Notice that $1_d \circ \epsilon_d = \epsilon_d \circ F(G(1_d))$. Also because F is a functor, $F(1_{G(d)}) = F(G(1_d))$. Because ϵ_d is an isomorphism, $F(1_{G(d)}) = F(G(1_d))$, and because F is faithful, $1_{G(d)} = G(1_d)$.
   For the composition, choose $f : d \to d' \in \mathsf{D}$ and $g : d' \to d''$. Notice that $(g \circ f) \circ \epsilon_d = \epsilon_{d''} \circ F(G(g \circ f))$. Now due to the

[37] These relations can be drawn as following.

$$\begin{array}{ccccccc} c & \xrightarrow{\eta_c} & GF(c) & \quad & d & \xrightarrow{\epsilon_d} & FG(d) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GF(f)} & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle FG(g)} \\ c' & \xrightarrow{\eta_{c'}} & GF(c') & & d' & \xrightarrow{\epsilon_{d'}} & FG(d) \end{array}$$

[38] This relation can be drawn as following.

$$\begin{array}{ccccc} c & \xrightarrow{\eta_c} & GF(c) & \xleftarrow{\eta_c} & c \\ {\scriptstyle f}\downarrow & & {\scriptstyle GF(f)}\downarrow{\scriptstyle GF(g)} & & \downarrow{\scriptstyle g} \\ c' & \xrightarrow{\eta_{c'}} & GF(c') & \xleftarrow{\eta_{c'}} & c' \end{array}$$

[39] Because we use the axiom of choice to construct the inverse functor G of F, if we are considering the categories with only the countably many objects, then we only need to use the countable choice of axiom, and if there are finitely many objects then we do not need any additional axiom.

[40] The relation can be drawn as following.

$$\begin{array}{ccc} F(G(d)) & \xrightarrow{\epsilon_d} & d \\ {\scriptstyle F(G(f))}\downarrow & & \downarrow{\scriptstyle f} \\ F(G(d')) & \xrightarrow{\epsilon_{d'}} & d' \end{array}$$

associativity and natural transformation-like property of ϵ, we have the following.

$$\begin{aligned}
\epsilon_{d''} \circ F(G(g \circ f)) &= (g \circ f) \circ \epsilon_d \\
&= g \circ (f \circ \epsilon_d) \\
&= g \circ (\epsilon_{d'} \circ F(G(f))) \\
&= (g \circ \epsilon_{d'}) \circ F(G(f)) \\
&= (\epsilon_{d''} \circ F(G(g))) \circ F(G(f)) \\
&= \epsilon_{d''} \circ (F(G(g)) \circ F(G(f))) \\
&= \epsilon_{d''} \circ (F(G(g) \circ G(f)))
\end{aligned}$$

Thus, because $\epsilon_{d''}$ is an isomorphism, we get $F(G(g \circ f)) = F(G(g) \circ G(f))$, and because F is faithful, $G(g \circ f) = G(g) \circ G(f)$. Finally, because ϵ is an isomorphic natural transformation, we can consider the map $\epsilon^{-1}_{F(c)} : F(c) \to FGF(c)$ for any object $c \in \mathsf{C}$. Because F is full, we have $\eta_c : c \to FG(c)$ satisfying $F(\eta_c) = \epsilon^{-1}_{F(c)}$. Now due to the definition of ϵ and η, for any $f : c \to c' \in \mathsf{C}$, we can consider the following.

$$\begin{aligned}
\epsilon_{F(c')} \circ FGF(f) \circ F(\eta_c) &= F(f) \circ \epsilon_{F(c)} \circ F(\eta_c) \\
&= F(f) \\
&= \epsilon_{F(c')} \circ F(\eta_{c'}) \circ F(f)
\end{aligned}$$

Because $\epsilon_{F(c')}$ is an isomorphism, we get $F(GF(f) \circ \eta_c) = F(\eta_{c'} \circ f$. Because F is faithful, $GF(f) \circ \eta_c = \eta_{c'} \circ f$, thus η is a natural transformation.

□

2021.01.13.

1.7 Duality and Opposite Category

| Definition 1.7.1 (ETAC). | The **atomic statement** in the **elementary theory of an abstract category(ETAC)** consists of:

1. the variables $a, b, c, \cdots$ for objects,
2. the variables $f, g, h, \cdots$ for arrows,
3. the letter dom for the domain,
4. the letter cod for the codomain,
5. the letter 1 for the identity,
6. the letter $\circ$ for the composition between composable arrows, and
7. the letter $=$ for the equality,

which are:

1. $a = b$,
2. $f = g$,
3. $a = \operatorname{dom} f$,

4. $b = \mathrm{cod}\, f$,
5. $g = 1$,
6. $h = g \circ f$.

A **statement** Σ is a well-formed phrase built up from the atomic statements above with connectives $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$ and quantifiers $\forall, \exists, \exists!, \nexists$.

A **sentence** is a statement with no free variables, that is, all the variables are quantified.[41]

[41] The axioms of abstract category in the Section 1.1 are all the sentences.

DEFINITION 1.7.2 (DUAL). Let Σ be a statement of ETAC. Then the **dual statement** of Σ, Σ^*, is a statement which changes all the atomic statements in the Σ as the following.

1. No change in $a = b$;
2. No change in $f = g$;[42]
3. Change $a = \mathrm{dom}\, f$ into $a = \mathrm{cod}\, f$;
4. Change $b = \mathrm{cod}\, f$ into $b = \mathrm{dom}\, f$;
5. No change in $g = 1$;
6. Change $h = g \circ f$ into $h = f \circ g$.[43]

[42] Indeed, we need to change each side of equality also, but due to the reflectivity of equality, it does not change.

[43] This change of the sequence of morphism is needed to make the composable pair between morphisms, whose domains and codomains are exchanged. See Proposition 1.7.4.

PROPOSITION 1.7.3. *For a statement Σ of ETAC, the dual of the dual is the original statement. In other words, $\Sigma = \Sigma^{**}$.*

Proof. The change 1, 2, and 5 are same. Changing 3, 4, and 6 twice gives the original statement. □

PROPOSITION 1.7.4. *For each axiom for a category, the dual of them is again an axiom.*

Proof. The existence of domain changes to the existence of codomain, and vice versa. The existence of identity morphism does not change under the dual. For the composability, which says $g \circ f$ is composable if and only if $\mathrm{cod}\, f = \mathrm{dom}\, g$, its dual statement becomes $f \circ g$ is composable if and only if $\mathrm{dom}\, f = \mathrm{cod}\, g$, and exchanging the letters f, g gives the desired result. □

PROPOSITION 1.7.5 (DUALITY PRINCIPLE). *If a statement Σ of ETAC is a consequence of the axioms, then so is the dual statement Σ^*.*

Proof. If we have the proof Π of the statement Σ, then the statement Π^* is the proof of the statement Σ^*. □

DEFINITION 1.7.6. For a category C, the **opposite category** C^{op} is a category, whose object is $\mathrm{ob}\,\mathsf{C}$, and morphisms are $f^{op} : y \to x$ for each $f : x \to y \in \mathsf{C}$. Here the identity on x is 1_x, and the composition rule becomes $g^{\mathrm{op}} f^{\mathrm{op}} = (fg)^{\mathrm{op}}$.

Corollary 1.7.7. *Suppose that Σ is a statement with free variables. Then Σ is true for some constant arrows $f, g, \cdots$ of a category C if and only if the dual statement Σ^* is true for some constant arrows $f^{op}, g^{op}, \cdots$ of a category C^{op}. Therefore, a sentence Σ is true in C if and only if a sentence Σ^* is true in C^{op}.*[44]

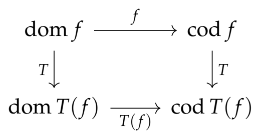
[44] To emphasize this property, sometimes we write C^{op} as C^*.

Proof. Suppose that the atomic sentences are all true under some constants a, b, f, g, h. If we change f into f^{op}, g into g^{op}, and h into h^{op}, then all the atomic sentences are true again. □

Example 1.7.8.

1. A map $T : \mathsf{C} \to \mathsf{D}$ is a functor if $\operatorname{dom} T(f) = T(\operatorname{dom} f)$, $\operatorname{cod} T(f) = T(\operatorname{cod} f)$, $T(1) = 1$, and $T(gf) = T(g)T(f)$ for all composable f, g.[45] Here, notice that f, g are bound variables. Now substitute f, g by the constants. Taking the dual on C and D gives the following:

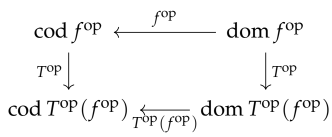
[45]



$$\begin{aligned} \operatorname{dom} T(f^{\mathrm{op}})^{\mathrm{op}} &= T(\operatorname{dom} f^{\mathrm{op}}) \\ \operatorname{cod} T(f^{\mathrm{op}})^{\mathrm{op}} &= T(\operatorname{cod} f^{\mathrm{op}}) \\ T(1)^{\mathrm{op}} &= 1 \\ T(f^{\mathrm{op}} g^{\mathrm{op}})^{\mathrm{op}} &= T(f^{\mathrm{op}})^{\mathrm{op}} T(g^{\mathrm{op}})^{\mathrm{op}} \end{aligned}$$

Define a functor $T^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$ as $T^{\mathrm{op}}(f^{\mathrm{op}}) = T(f^{\mathrm{op}})^{\mathrm{op}}$ by the following data: [46]

[46]



$$\begin{aligned} \operatorname{dom} T^{\mathrm{op}}(f^{\mathrm{op}}) &= T^{\mathrm{op}}(\operatorname{dom} f^{\mathrm{op}}) \\ \operatorname{cod} T^{\mathrm{op}}(f^{\mathrm{op}}) &= T^{\mathrm{op}}(\operatorname{cod} f^{\mathrm{op}}) \\ T^{\mathrm{op}}(1) &= 1 \\ T^{\mathrm{op}}(f^{\mathrm{op}} g^{\mathrm{op}}) &= T^{\mathrm{op}}(f^{\mathrm{op}}) T^{\mathrm{op}}(g^{\mathrm{op}}) \end{aligned}$$

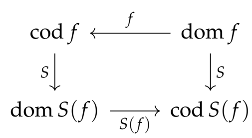
This is exactly same with above condition. We call T^{op} the **dual functor**.

2. Now, take only the dual on C, not on D. Then we get the following:

$$\begin{aligned} \operatorname{dom} T(f^{\mathrm{op}}) &= T(\operatorname{cod} f^{\mathrm{op}}) \\ \operatorname{cod} T(f^{\mathrm{op}}) &= T(\operatorname{dom} f^{\mathrm{op}}) \\ T(1) &= 1 \\ T(f^{\mathrm{op}} g^{\mathrm{op}}) &= T(g^{\mathrm{op}}) T(f^{\mathrm{op}}) \end{aligned}$$

This is again a functor. Define a map $S : \mathsf{C} \to \mathsf{D}$ by the following data:[47]

[47]



$$\begin{aligned} \operatorname{dom} S(f) &= S(\operatorname{cod} f) \\ \operatorname{cod} S(f) &= S(\operatorname{dom} f) \\ S(1) &= 1 \\ S(fg) &= S(g)S(f) \end{aligned}$$

This is just a renaming of f^{op} to f, g^{op} to g, and C^{op} to C. We call T a **contravariant functor** on C to D. The functor defined originally is called a **covariant functor** from C to D.[48]

Chapter 2

Special Objects, Morphisms, Functors and Categories

You're so fuckin' special.
— Radiohead, *Creep*

2.1 Hom-functor and Initial, Final, Zero Object

Definition 2.1.1 (Hom-functor). Consider an object $c \in \mathsf{C}$. We call $\hom(c, -) : \mathsf{C} \to \mathsf{Set}$ as a **covariant hom-functor** and $\hom(-, c) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ as a **contravariant hom-functor**. Here, for $f : d \to e$, $\hom(c, f) = f_*$ is a post-composition, and $\hom(f, c) = f^*$ is a pre-composition.

Definition 2.1.2 (Constant functor). A functor $* : \mathsf{C} \to \mathsf{Set}$ is called a **constant functor** if $*(c) = \{\bullet\}$ is a singleton set for all $c \in \mathsf{C}$.[1]

[1] Here, all the morphisms becomes the identity morphism on a singleton set, which is the only morphism.

Definition 2.1.3 (Initial, Final, and Zero object). Consider a category C.

1. An object $s \in \mathsf{C}$ is an **initial object** if for any object $c \in \mathsf{C}$, there is exactly one morphism $s \to c$ in $\hom(s, c)$.
2. An object $t \in \mathsf{C}$ is a **final object** if for any object $d \in \mathsf{C}$, there is exactly one morphism $d \to t$ in $\hom(d, t)$.
3. An object 0 is a **zero object** an object which is both initial and terminal.

Proposition 2.1.4. *If a category C has a initial, terminal, or zero object, then it has only one initial, terminal, or zero object respectively, up to isomorphism.*

Proof. Suppose that t, t' are both terminal objects. Then the arrows in $t \to t' \to t \to t'$ are unique, whose compositions becomes a unique endomorphism on t, t', which is 1_t and $1'_t$. Hence t and t' are isomorphic.

Dually, initial object is unique.

Because a zero object is terminal object, it is unique. □

Example 2.1.5.

1. For a category Set, the empty set ϕ is an initial object, and the singleton set $\{\bullet\}$ is a terminal object. Because they are not isomorphic, there is no zero element in Set.
2. For a category Group, the singleton group 0 is both an initial object and final object, hence a zero object. This is same under Ring and Mod_R.
3. Consider a two-object category C, with two parallel non-identity morphisms. Then there is no initial and final object, hence no zero object.

Proposition 2.1.6. *Consider an object $c \in \mathsf{C}$.*

1. *c is initial if and only if the covariant functor* $\hom(c, -) : \mathsf{C} \to \mathsf{Set}$ *is naturally isomorphic to the constant functor* $* : \mathsf{C} \to \mathsf{Set}$.
2. *c is final if and only if the contravariant functor* $\hom(-, c) : \mathsf{C}^{op} \to \mathsf{Set}$ *is naturally isomorphic to the constant functor* $* : \mathsf{C}^{op} \to \mathsf{Set}$.

Proof. Because if c is initial in C if and only if c is final in C^{op}, the statements above are in dual relation, hence we only need to show the first relation.

1. Define $\eta : \hom(c, -) \Rightarrow *$ as $\eta_d : \hom(c, d) \to 1$ and $\epsilon : * \Rightarrow \hom(c, -)$ as $\epsilon_d : 1 \to \hom(c, d)$. If c is initial then η, ϵ are natural isomorphisms, and conversely if they are natural isomorphisms then $\hom(c, d)$ is a singleton set for all object $d \in \mathsf{C}$.

□

2021.01.19.

2.2 *Zero Morphism*

Definition 2.2.1 ((co-)Constant morphism). Let $f \in \mathsf{C}$ be a morphism.

1. If $fg = fh$ for any composable morphisms $g, h \in \mathsf{C}$, we call f a **constant morphism** or **left zero morphism**.
2. Dually, if $gf = hf$ for any composable morphisms $g, h \in \mathsf{C}$, we call f a **coconstant morphism** or **right zero morphism**.
3. If f is constant morphism and coconstant morphism, then we call f a **zero morphism**.

Proposition 2.2.2. *A composition between a zero morphism and any morphism is a zero morphism.*

Proof. Let 0 be a zero morphism. Consider $0f$ for a composable morphism f. Then $(0f)g = 0(fg) = 0(fh) = (0f)h$ for any composable morphisms g, h, hence $0f$ is a zero morphism. Dually, $j0$ is a zero morphism for a composable morphism j. □

Example 2.2.3.

1. In the category Group and Mod_R, a zero morphism is a homomorphism mapping all the elements to the identity element 1. Thus, every morphism $f : X \to Y$ in Group or Mod_R can be decomposed as $f : X \to 1 \to Y$.
2. If we consider a category with two objects and two nontrivial parallel morphisms, then both two morphisms are vacuously zero morphisms. Hence zero morphism is noy always unique.

Definition 2.2.4. Let C be a category. If every hom-set $\hom(c, d)$ contains a zero morphism 0_{cd}, and these zero morphisms satisfies $0_{cd}f = 0_{bd}$ and $f0_{ab} = 0_{ac}$, for every $a, b, c, d \in \mathsf{C}$ and $f : b \to c \in \mathsf{C}$, then we call C a **category with zero morphisms**.

Proposition 2.2.5. *If a category* C *has a zero object, then* C *is a category with zero morphisms.*

Proof. For each objects $c, d \in \mathsf{C}$, define a map $0_{cd} : c \to 0 \to d$, where 0 is a zero object, which is well defined. Consider $f, g : b \to c$. Then $0_{cd} \circ f : b \to c \to 0 \to d = b \to 0 \to d = 0_{bd}$ is equal to $0_{cd} \circ g$, hence 0_{cd} is a constant morphism. Dually, 0_{cd} is a coconstant morphism, hence a zero morphism. □

Proposition 2.2.6. *If a category* C *is a category with zero morphisms, with the collection of zero morphisms* $\{0_{bc} \in \hom(b, c) : b, c \in \mathsf{C}\}$*, then this collection of zero morphisms is unique.*

Proof. Suppose that we have another collection $\{0'_{bc}\}$. Then because $0_{bc} = 0_{cc}0_{bc} = 0_{cc}0'_{bc} = 0'_{cc}0'_{bc} = 0'_{bc}$ for any $b, c \in \mathsf{C}$, we get the desired result. □

2021.01.20.

2.3 Groupoid, Connected Category, and Skeletal Category

Definition 2.3.1 (Groupoid). A **groupoid** is a category in which every morphism is an isomorphism.

Example 2.3.2.

1. A **discrete category**, which is a category without nonidentity morphisms, is a groupoid.
2. Let G be a group. Then the category $\mathsf{B}G$ is a groupoid.
3. Let C be a category. Then there is a unique **maximal groupoid**, which is the subcategory containing all of the objects and only isomorphic morphisms.[2]
4. Let X be a space. Then its **fundamental groupoid** $\Pi_1(X)$ is a category, whose objects are the points of X and whose morphisms are endpoint-preserving homotopy classes of paths.

[2] Because the composition of isomorphisms is isomorphism, this is indeed a subcategory.

Lemma 2.3.3. *A morphism* C *is an isomorphism if and only if its image under an equivalence* $\mathsf{C} \xrightarrow{\sim} \mathsf{D}$ *is an isomorphism.*

Proof. Let functors $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be an equivalence of categories. Then by the Theorem 1.6.3, F and G are fully faithful. Consider a morphism $f \in \mathsf{C}$. Then $F(f) \in \mathsf{D}$ is an isomorphism, thus has an inverse g. Because F is full, there is $f' \in \mathsf{C}$ such that $g = F(f')$. Thus $F(ff')$ and $F(f'f)$ are identities. Because F is faithful, ff' and $f'f$ are identities. Hence f is an isomorphism.

Conversely, let f be an isomorphism, whith inverse f'. Then $F(f)F(f')$ and $F(f')F(f)$ are identities, due to the functor property. □

Proposition 2.3.4. *Let* C *be a groupoid, and* $\mathsf{C} \simeq \mathsf{D}$. *Then* D *is also a groupoid.*

Proof. The image of every $f \in \mathsf{D}$ under equivalence is isomorphic. Due to the Lemma 2.3.3, it implies that f is isomorphic. Hence D is a groupoid. □

Proposition 2.3.5. *An opposite category of a groupoid* C *is equivalent to* C.

Proof. Define $F : \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ as $F(f) = (f^{-1})^{\mathrm{op}}$. This map is fully faithful and essentially surjective on objects, hence by the Theorem 1.6.3, F is an equivalence of cagtegories. □

Definition 2.3.6 (Connected category). A category is **connected** if it is not empty, and any pair of objects can be connected by a finite composition of morphisms.

Proposition 2.3.7. *Any connected groupoid is equivalent to the automorphism group of any of its objects.*

Proof. For a connected groupoid G, choose an object $g \in \mathsf{G}$, and let $G = \hom(g, g)$ be its automorphism group. The inclusion $\mathsf{B}G \hookrightarrow \mathsf{G}$ is then, by definition, fully faithful and essentially surjective on objects[3], hence G is equivalence. $\square$

[3] To show the essential surjectivity, we need the connectedness.

| Corollary 2.3.8. | *Let X be a path-connected space. Then any choice of basepoint $x \in X$ gives an isomorphic fundamental group $\pi_1(X, x) = \pi_1(X)$.*

Proof. Because X is path connected, the fundamental groupoid $\Pi_1(X)$ is connected. Also, $\pi_1(X, x)$ is an automorphism group of the object $x\Pi_1(X)$. Thus by the proposition 2.3.7, $\pi_1(X, x) \simeq \Pi_1(X) \simeq \pi_1(X, x')$ for any points $x, x' \in X$, thus $\pi_1(X, x) \simeq \pi_1(X, x')$ categorically, which is also an isomorphism of groups. $\square$

| Definition 2.3.9 (Skeletal category). | A category C is called a **skeletal category** if there is only one object in each isomorphism class.

| Lemma 2.3.10. | *Let C and D be skeletal categories. If C and D are equivalent, then they are isomorphic.*

Proof. Suppose that $F : \mathsf{C} \to \mathsf{D}$ be an equivalence of categories. Because F is fully faithful and essentially surjective on objects, and D is skeletal, F is bijective on morphisms and surjective on objects. Suppose that we have $c, c' \in \mathsf{C}$ such that $F(c) = F(c') = d$. Then due to the fullness, we have $f : c \to c'$ and $g : c' \to c$ such that $F(f) = F(g) = 1_d$. Then $F(fg) = F(gf) = 1_d$, thus by faithfulness, $fg = 1_{c'}$ and $gf = 1_c$, showing that $c \simeq c'$. Because C is skeletal, $c = c'$. Hence F is bijective on objects. $\square$

| Proposition 2.3.11. | *Let Axiom of Choice be true. For a nonempty category C, there is a* ***skeleton*** $\mathrm{sk}\mathsf{C}$ *of a category C, which is the (up to isomorphism) unique skeletal category equivalent to C.*

Proof. From each isomorphism class of C, choose an object, using Axiom of Choice. For each morphisms $f : b \to c \in \mathsf{C}$, define a morphism $f' : b' \to c'$, where b' and c' are the chosen objects of isomorphism classes containing b and c, respectively. For $f : b \to c$ and $g : d \to e$ in C, we say g' and f' are composable if and only if $c \simeq d$, and $g'f'$ as gif, where $i : c \to d$ is an isomorphism. From these data, define $\mathrm{sk}\mathsf{C}$. By the definition, the inclusion $\mathrm{sk}\mathsf{C} \hookrightarrow \mathsf{C}$ is fully faithful and essentially surjective on objects, hence equivalence. By the Lemma 2.3.10, every skeleton of C are isomorphic. $\square$

2.4 Comma Category

Definition 2.4.1 (Comma category). Let C, D, and E be categories, and $F : \mathsf{E} \to \mathsf{C}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors. The **comma category** $(F \downarrow G)$ is a category defined with following data:[4]

1. Objects are all triples
$$(e, d, f : F(e) \to G(d)) \in \operatorname{ob} \mathsf{E} \times \operatorname{ob} \mathsf{D} \times \hom_{\mathsf{C}}(F(e), G(d)) \quad (2.1)$$
and,
2. Arrows $(e, d, f) \to (e', d', f')$ are all pairs
$$(k : e \to e', h : d \to d') \in \hom_{\mathsf{E}}(e, e') \times \hom_{\mathsf{D}}(d, d') \quad (2.2)$$
satisfying $f' \circ F(k) = G(h) \circ f$.
3. The composition $(k', h') \circ (k, h)$ is defined as $(k' \circ k, h' \circ h)$.

[4]
$$\begin{array}{ccc} F(e) & \xrightarrow{F(k)} & F(e') \\ f\downarrow & & \downarrow f' \\ G(d) & \xrightarrow{G(h)} & G(d') \end{array}$$

Proposition 2.4.2. *Let $a, b \in \mathsf{C}$ be objects. Abusing the notation, we define a functor $a : 1 \to \mathsf{C}$ whose image is a,[5] and same with b. Then $(a \downarrow b)$ is equivalent to the discrete category* $\hom(a, b)$.

[5] This notation is frequently used. If there is an object where we need to put a functor, then this object is considered as such functor.

Proof. The objects of $(a \downarrow b)$ are $(\bullet, \bullet, f : a \to b)$, where $\bullet$ is the object of category 1. We may write $(\bullet, \bullet, f : a \to b)$ as f.

The morphisms of $(a \downarrow b)$ from f to g is a pair of morphisms $k, h : \bullet \to \bullet$ satisfying $ga(k) = b(h)f$. Because $a(k) = 1_a$ and $b(h) = 1_b$, $g = f$. Thus the only morphisms in $(a \downarrow b)$ are identities. □

Definition 2.4.3 (Slice categories). Let $c \in \mathsf{C}$ be an object.

1. We define a **slice category under** c as $(c \downarrow \mathsf{C})$, and write it c/C.
2. We define a **slice category over** c as $(\mathsf{C} \downarrow c)$, and write it C/c.

2.5 Functor Category

2021.01.21.

Definition 2.5.1 (Functor category). For the categories B and C, we define a **functor category** $\mathsf{B}^{\mathsf{C}} = \operatorname{Funct}(\mathsf{C}, \mathsf{B})$ with following data:

- Objects are all functors $T : \mathsf{C} \to \mathsf{B}$, and,
- Arrows $S \to T$ are all natural transformations $S \Rightarrow T \in \operatorname{Nat}(S, T)$.
- The composition $\epsilon \circ \eta$ is defined as the usual composition of natural transformations.

Example 2.5.2. 1. For small categories B and C, B^{C} is also a small category.
2. For a small discrete category C, $\{0, 1\}^{\mathsf{C}}$ is isomorphic to the set of all subsets of C.

2.6 The category of categories

Definition 2.6.1. The category Cat is a category with following data:

- Objects are the small cataegories C;
- Arrows are all functors $F : \mathsf{C} \to \mathsf{D}$.
- The composition $G \circ F$ is defined as the usual composition of functors.

Proposition 2.6.2. *The category* Cat *is a locally small category.*

Proof. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between small categories. On objects, this functor have at most $\operatorname{ob} \mathsf{D}^{\operatorname{ob} \mathsf{C}}$-many choices, and for each choices, there are at most $\operatorname{mor} \mathsf{D}^{\operatorname{mor} \mathsf{C}}$-many choices. Thus in total there are at most $\operatorname{ob} \mathsf{D}^{\operatorname{ob} \mathsf{C}} \times \operatorname{mor} \mathsf{D}^{\mathsf{C}}$-many functors, which is a set. □

Definition 2.6.3. In Cat, there exists a **product** between two categories C and D, defined as following:

- Objects are all pairs (c, d) of objects;
- Arrows $(c, d) \to (c', d')$ are all pairs $(f : c \to c', g : d \to d')$;
- The composition is defined as $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$.

We write this category $\mathsf{C} \times \mathsf{D}$.

The functors $P : \mathsf{C} \times \mathsf{D} \to \mathsf{C}$ and $Q : \mathsf{C} \times \mathsf{D} \to \mathsf{D}$, defined as

$$P(f, g) = f, \quad Q(f, g) = g \tag{2.3}$$

are called the **projections**.

Definition 2.6.4. A functor $F : \mathsf{C} \times \mathsf{D} \to \mathsf{B}$ is called a **bifunctor**.

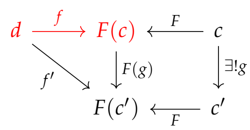
Chapter 3

Universality

Come, said the Muse,
Sing me a song no poet yet has chanted,
Sing me the Universal.
— Ola Gjeilo, *Song of the Universal*

3.1 *Universal Object and Morphism*

Definition 3.1.1 (Universal morphism). Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Then a **universal morphism** from $d \in \mathsf{D}$ to F is a pair $(c, f : d \to F(c)) \in \operatorname{ob} \mathsf{C} \times \hom_{\mathsf{D}}(d, F(c))$, such that for any $(c', f' : d \to F(c')) \in \operatorname{ob} \mathsf{C} \times \hom_{\mathsf{D}}(d, F(c'))$, there is a unique arrow $g : c \to c' \in \mathsf{C}$ satisfying $F(g) \circ f = f'$.[1]

[1] 

Example 3.1.2.

1. Consider the forgetful functor $U : \mathsf{Vec}_k \to \mathsf{Set}$, and a set X. Then the universal morphism from X to U is a pair $(V_X, j : X \hookrightarrow U(V_X))$, where V_X is a vector space with basis X. Indeed, consider $f : X \to U(W)$ for some vector space W. Then we can find a unique morphism $g : V_X \to W$ such that $F(g) \circ j = f$, where the map g is defined from the basis $\{f(x) : x \in X\}$.
2. Indeed, there is a universal morphism for each well-known forgetful functor. For example, consider the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$. Then a universal morphism from a set X to U is a **free group** $F(X)$ and inclusion $j : X \hookrightarrow F(X)$. Similarly, $U : \mathsf{Ring} \to \mathsf{Set}$ gives a **free ring**, $U : \mathsf{Mod}_R \hookrightarrow \mathsf{Set}$ gives a **free module**, and so on.
3. Let Met be a category of all metric spaces with metric-preserving morphisms. Then the category CMet, a category of complete metric spaces, is a full subcategory.
   Now we consider the forgetful functor $U : \mathsf{CMet} \to \mathsf{Met}$. Then a universal morphism from a metric space X to U is a map $j : X \hookrightarrow \overline{X}$, where $\overline{X}$ is a completion of X.

Proposition 3.1.3. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor and $d \in \mathsf{D}$ be an object. Then $(c, f : d \to F(c))$ is a universal from d to F if and only if (c, f) is an initial object in the comma category $(d \downarrow F)$.*[2]

Proof. The object $(\bullet, c, f : d \to F(c))$ is an initial object if and only if, for any other objects $(\bullet, c', f' : d \to F(c'))$, there is a unique morphism $(\bullet, g : c \to c')$ satisfying $F(g) \circ f = f'$, which is the definition of universal morphism. □

[2] Therefore, because the initial object is unique up to isomorphism, if (c, f) is a universal morphism, then it is unique up to isomorphism.

Definition 3.1.4 (Universal object). Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor. Then a **universal element** of the functor F is a pair $(c, x \in F(c))$ such that for every pair $(d, y \in F(d))$, there is a unique morphism $f : c \to d$ satisfying $F(f)(x) = y$.

Proposition 3.1.5.

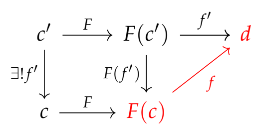
1. *Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor. Then $(c, x \in F(c))$ is a universal element if and only if $(c, x : \{\bullet\} \to F(c))$ is a universal morphism from $\{\bullet\}$ to F.*
2. *Let D be a locally small category, and $F : \mathsf{C} \to \mathsf{D}$ be a functor with $d \in \mathsf{D}$ be an object. Then $(c, f : d \to F(c))$ is a universal arrow from d to F if and only if the pair $(c, f \in \mathsf{D}(d, F(c)))$ is a universal element of the functor $G = \mathsf{D}(d, F(-))$.*

Proof. This directly follows from the definition. □

3.1.D Universal Object and Morphism: Dual

Definition 3.1.6 (Dual Universal morphism). Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Then a **universal morphism** from F to $d \in \mathsf{D}$ is a pair $(c, f : F(c) \to d \in \operatorname{ob}\mathsf{C} \times \hom_{\mathsf{D}}(F(c), d)$, such that for any $(c', f' : F(c') \to d \in \operatorname{ob}\mathsf{C} \times \hom_{\mathsf{D}}(F(c'), d)$, there is a unique arrow $g : c' \to c \in \mathsf{C}$ satisfying $f \circ F(g) = f'$.[3]

[3]



Proposition 3.1.7. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor and $d \in \mathsf{D}$ be an object. Then $(c, f : F(c) \to d)$ is a universal from F to d if and only if (c, f) is a terminal object in the comma category $(F \downarrow d)$.*

Proof. This can be proven by the dual proof of 3.1.3. □

2021.01.22.

3.2 Representation of a Functor

Definition 3.2.1. A representation of a functor $F : \mathsf{C} \to \mathsf{Set}$ is a pair (c, η), where $c \in \mathsf{C}$ is an object and $\eta : \mathsf{C}(c, -) \simeq F$ is a natural isomorphism. We say c a **representing object**. If such a representation exists, then we call F a **representable functor**.

Proposition 3.2.2. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor and $c \in \mathsf{C}$, $d \in \mathsf{D}$ be objects.*

1. *A pair $(c, f : d \to F(c))$ is a universal morphism from d to F if and only if, for any object $c' \in \mathsf{C}$, we have a bijection of hom-sets,*
$$\mathsf{C}(c, c') \simeq \mathsf{D}(d, F(c')) \tag{3.1}$$
which takes each $f' : c \to c'$ into $F(f') \circ f : d \to F(c')$.
2. *The bijection in equation 3.1 is natural in c', that is, we have a natural isomorphism $\mathsf{C}(c, -) \simeq \mathsf{D}(d, F(-))$.*
3. *Conversely, any natural isomorphism between $\mathsf{C}(c, -) \simeq \mathsf{D}(d, F(-))$ is determined by a unique arrow $f : d \to F(c)$ such that $(c, f : d \to F(c))$ is a universal morphism from d to F.*

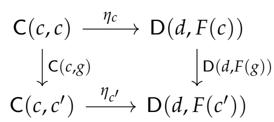
Thus, c represents $\mathsf{D}(d, F(-))$.

Proof.

1. The definition of universality directly implies the bijection.
2. Let $f' : c \to c'$ and $g : c' \to c''$. Then $F(gf') \circ f = F(g)(F(f') \circ f)$.
3. Take a natural isomorphism $\eta : \mathsf{C}(c, -) \simeq \mathsf{D}(d, F(-))$. Then for each $c' \in \mathsf{C}$, we get a bijection $\eta_{c'} : \mathsf{C}(c, c') \to \mathsf{D}(d, F(c'))$. Then for any $g : c \to c'$, due to the naturality, we get $\eta_{c'} \circ \mathsf{C}(c, g) = \mathsf{D}(d, F(g)) \circ \eta_c$.[4] Now putting $1_c \in \mathsf{C}$ gives $\eta_{c'}(g)$ and $F(g) \circ \eta_c(1_c)$, for left and right side respectively. Now define $f : d \to F(c)$ as $\eta_c(1_c)$, then we get $\eta_{c'}(g) = F(g) \circ f$. Because $\eta_{c'}$ is an isomorphism, we get the result that for each $f' = \eta_{c'}(g)$, there is a unique g satisfying $f' = F(g) \circ f$, which shows that (c, f) is a universal morphism.

□

4



3.2.D Representation of a Functor: Dual

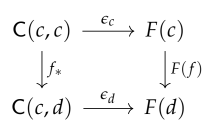
Definition 3.2.3. A representation of a functor $F^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is a pair (c, η), where $c \in \mathsf{C}$ is an object and $\eta : \mathsf{C}(-, c) \simeq F$ is a natural isomorphism. We say c a **representing object**. If such a representation exists, then we call F a **representable functor**.

2021.01.27.

3.3 The Yoneda Lemma

Lemma 3.3.1 (Yoneda lemma). *Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor and $c \in \mathsf{C}$ be an object. Then there is a bijection*
$$\mathrm{Nat}(\mathsf{C}(c, -), F) \simeq F(c) \tag{3.2}$$
which sends each natural transformation $\epsilon : \mathsf{C}(c, -) \Rightarrow F$ to $\epsilon_c(1_c)$.[5]

5



Proof. In the Proposition 3.2.2, take D as Set and $d := \{\bullet\}$. Then we only need to show that there is a natural isomorphism η between $\mathsf{Set}(\{\bullet\}, F(-))$ and F. But because $\eta_c : \mathsf{Set}(\{\bullet\}, F(c)) \to F(c)$ defined by the image of $\bullet \to F(c)$ gives a natural isomorphism, we get the desired result. □

Corollary 3.3.2. *Let $c, d \in \mathsf{C}$. Then each natural transformation $\epsilon : \mathsf{C}(c, -) \Rightarrow \mathsf{C}(d, -)$ has the form $\mathsf{C}(d \to c, -)$ for a unique arrow $d \to c$. Furthermore, if ϵ is an isomorphism, then the arrow $d \to c$ is also an isomorphism.*

Proof. By the Lemma 3.3.1, each natural transformation $\epsilon : \mathsf{C}(c, -) \Rightarrow \mathsf{C}(d, -)$ are related to morphisms $h : d \to c$. Because this image is the result of $\epsilon_c 1_c$, we get $\epsilon := \mathsf{C}(h, -)$.

If ϵ is isomorphism then there is its inverse $\eta : \mathsf{C}(d, -) \Rightarrow \mathsf{C}(c, -)$. Suppose that ϵ is related to h and η is related to k. Now $\eta \circ \epsilon = 1_{\mathsf{C}(c,-)}$ must have the form $\mathsf{C}(1_c, -) = \mathsf{C}$. Due to the uniqueness, $h \circ k = 1_c$. Similarly, $k \circ h = 1_d$. □

Lemma 3.3.3 (Naturality of Yoneda Lemma). *Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor. The bijection*

$$y : \mathrm{Nat}(\mathsf{C}(c, -), F) \simeq F(c) \tag{3.3}$$

is a natural isomorphism $\epsilon : N \Rightarrow E$ between the functors $N, E : \mathsf{Set}^{\mathsf{D}} \times \mathsf{D} \to \mathsf{Set}$.[6]

[6] The functor N here means the natural morphism functor, taking (F, c) to $\mathrm{Nat}(\mathsf{C}(c, -), F)$, and the functor E mean the evaluation functor $E(F, c) = F(c)$.

Proof. For the naturality on functor, let $\epsilon : F \Rightarrow G$. Now take a natural transformation $\alpha \in \mathrm{Nat}(\mathsf{C}(c, -), F)$. Then by β, it becomes $\beta\alpha \in \mathrm{Nat}(\mathsf{C}(c, -), G)$, and then $(\beta\alpha)_c 1_c \in G(c)$ by Yoneda lemma. Also, α becomes $\alpha_c 1_c \in F(c)$ by Yoneda lemma, and then $\beta_c(\alpha_c 1_c)$ by β. Because these two results are same, y is natural on functor[7].

For the naturality on object, let $f : c \to d \in \mathsf{C}$. Now take a natural transformation $\alpha \in \mathrm{Nat}(\mathsf{C}(c, -), F)$. Then by f, it becomes αf^*, and by Yoneda lemma, $(\alpha f^*)_d(1_d) = \alpha_d(f)$. Also by Yoneda lemma, α becomes $\alpha_c(1_c)$, and by f, we get $F(f)(\alpha_c(1_c)) = \alpha_d(f)$, by Yoneda lemma.[8] □

[7]

$$\begin{array}{ccc} \mathrm{Nat}(\mathsf{C}(c,-),F) & \xrightarrow{y_F} & F(c) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_c} \\ \mathrm{Nat}(\mathsf{C}(c,-),G) & \xrightarrow{y_G} & G(c) \end{array}$$

[8]

$$\begin{array}{ccc} \mathrm{Nat}(\mathsf{C}(c,-),F) & \xrightarrow{y_c} & F(c) \\ \downarrow{\scriptstyle (f^*)^*} & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Nat}(\mathsf{C}(d,-),F) & \xrightarrow{y_d} & F(d) \end{array}$$

See the sidenote 5 in this section also.

Definition 3.3.4. Let $Y_{\mathsf{D}^{\mathrm{op}}} : \mathsf{D}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{D}}$ be a functor defined by the following data:

- $Y_{\mathsf{D}^{\mathrm{op}}} : d \mapsto \mathsf{D}(d, -)$ on object;
- $Y_{\mathsf{D}^{\mathrm{op}}} : (f : c \to d) \mapsto (\mathsf{D}(f, -) : \mathsf{D}(d, -) \Rightarrow \mathsf{D}(c, -))$ on arrow.

Then this is a faithful functor[9], called the **Yoneda functor**.

[9] By the Lemma 3.3.1. Because Y is also injective on objects, it is also called a **Yoneda embedding**.

Corollary 3.3.5 (Cayley's theorem). *Any group is isomorphic to a subgroup of a permutation group.*[10]

[10] Hence, sometimes we say the Yoneda lemma as the generalization of Cayley's theorem.

Proof. For a group G, consider the category BG. Then the Yoneda functor $BG \to \mathsf{Set}^{BG^{op}}$ gives the isomorphism between the set of morphisms $g \in BG$ and the set of natural transformations $\mathsf{D}(g,-) : \mathsf{D}(\bullet,-) \Rightarrow \mathsf{D}(\bullet,-)$, each are described by a morphism $\mathsf{D}(g,\bullet)\mathsf{D}(\bullet,\bullet) \to \mathsf{D}(\bullet,\bullet)$, which is the right multiplication. Because all these morphisms are distinct isomorphisms, G is a subgroup of the automorphism group on the set $\mathsf{D}(\bullet,\bullet)$. □

3.3.D
The Yoneda Lemma: Dual

Lemma 3.3.6. *Let $F : \mathsf{C}^{op} \to \mathsf{Set}$ be a functor and $c \in \mathsf{C}$ be an object. Then there is a bijection*

$$\mathrm{Nat}(\mathsf{C}(-,c),F) \simeq F(c) \tag{3.4}$$

which sends each natural transformation $\epsilon : \mathsf{C}(-,c) \Rightarrow F$ to $\epsilon_c(1_c)$.

Proof. The dual of the proof of the Lemma 3.3.1 shows this statement. □

Definition 3.3.7. Let $Y_{\mathsf{D}} : \mathsf{D} \to \mathsf{Set}^{\mathsf{D}^{op}}$ be a functor defined by the following data:

- $Y_{\mathsf{D}} : d \mapsto \mathsf{D}(-,d)$ on object;
- $Y_{\mathsf{D}} : (f : c \to d) \mapsto (\mathsf{D}(-,f) : \mathsf{D}(-,c) \Rightarrow \mathsf{D}(-,d))$ on arrow.

Then this is a faithful functor, called the **dual Yoneda functor**, or just the **Yoneda functor**.

3.4
Category of Elements

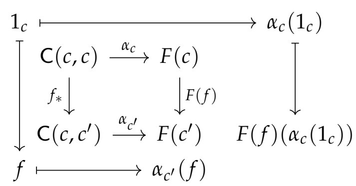
Definition 3.4.1. Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor. Then the category $(\{\bullet\} \downarrow F)$ is called the **category of elements**, and written as $\int^{\mathsf{C}} F$.[11]

[11] This notation comes from the concept called **coend**.

Proposition 3.4.2. *Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor. Then the category of elements $\int^{\mathsf{C}} F$ is isomorphic to the comma category $(Y \downarrow F)$, where $Y : \mathsf{C}^{op} \to \mathsf{Set}^{\mathsf{C}}$ is the Yoneda functor and $F : 1 \to \mathsf{Set}^{\mathsf{C}}$.*

Proof. The objects in $\int^{\mathsf{C}} F$ are $(c, x \in F(c))$, and the objects in $(Y \downarrow F)$ are $(\mathsf{C}(c,-), \alpha : \mathsf{C}(c,-) \Rightarrow F)$, which are bijective by Lemma 3.3.1 with $x \mapsto \alpha_c(1_c)$. Now notice that the morphism $f : c \to c'$ with $Ff(x) \mapsto Ff(x')$ uniquely defines the morphism between $\alpha : \mathsf{C}(c,-) \Rightarrow F$ and $\beta : \mathsf{C}(c',-) \Rightarrow F$ as $\mathsf{C}(f^{op},-) : \mathsf{C}(c',-) \Rightarrow \mathsf{C}(c,-)$ with $\mathsf{C}(f^{op},c')(1_{c'}) = f \in \mathsf{C}(c,c')$ which satisfies $\alpha \circ \mathsf{C}(f^{op},-) = \beta$. But it implies $\alpha_{c'} \circ \mathsf{C}(f^{op},c')(1_{c'}) = \alpha_{c'}(f) = \beta_{c'}(1_{c'}) \in F(c')$. Now, $F(f)(\alpha_c(1_c)) = \alpha_{c'}(f)$ due to the property of natural transformation[12], the morphism between α and β also has gives the unique morphism $f : c \to c'$ with $F(f)(\alpha_c(1_c)) \mapsto F(f)(\beta_{c'}(1_{c'}))$. □

[12]



Theorem 3.4.3. *Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor. Then F is representable if and only if $\int^{\mathsf{C}} F$ has an initial object.*[13]

[13] Hence the representation of a functor is unique up to isomorphism.

Proof. By Proposition 3.4.2, F is representable if and only if a natural isomorphism $\alpha : \mathsf{C}(c, -) \Rightarrow F$ exists as an object in $(Y \downarrow F)$, which is initial because the morphism from α to $\beta : \mathsf{C}(c', -) \Rightarrow F$ is defined uniquely by $\alpha^{-1} \circ \beta : \mathsf{C}(c', -) \Rightarrow \mathsf{C}(c, -)$. $\square$

Chapter 4

Limits

We've temporarily limited some of your account features.
— Twitter, *Donald Trump Jr.'s Limited account*

4.1
Limits on Set

| Definition 4.1.1. | Let I be a small category and C be a category. Then we say $\alpha : I \to \mathsf{C}$ an **inductive system**. Dually, we say $\beta : I^{\mathrm{op}} \to \mathsf{C}$ a **projective system**.

| Definition 4.1.2. | Let $\beta : I^{\mathrm{op}} \to \mathsf{Set}$ b a projective system. Then the **projective limit** of β is defined as the following.

$$\varprojlim_I \beta = \mathrm{Nat}(\{\bullet\}, \beta) \tag{4.1}$$

Here, $\{\bullet\} : I^{\mathrm{op}} \to \mathsf{Set}$ is a single point set constant functor.

| Proposition 4.1.3. | *For a projective system $\beta : I^{op} \to \mathsf{Set}$, the following holds.*[1]

$$\varprojlim_I \beta \simeq \left\{ \{x_i\}_{i \in I} \in \prod_{i \in I} \beta(i) : \beta(s)(x_j) = x_i, \forall s \in I(i,j) \right\} \tag{4.2}$$

Proof. Notice that the natural isomorphism $x : \{\bullet\} \Rightarrow \beta$ is described by the elements $x_i \in \beta(i)$ and morphisms $\beta(s) : \beta(j) \to \beta(i)$ with $x_j \mapsto x_i$.[2] This is the described set above. □

| Lemma 4.1.4. | *Let $\beta : I^{op} \to \mathsf{Set}$ be a functor and $X \in \mathsf{Set}$ be a set. Then there is a following natural isomorphism.*[3]

$$\mathsf{Set}(X, \varprojlim_I \beta) \simeq \varprojlim_I \mathsf{Set}(X, \beta) \tag{4.3}$$

[1] Hence, $\varprojlim_I \beta$ is a small set.

[2]
$$\begin{array}{ccc} \bullet & \xrightarrow{x_j} & \beta(j) \\ \| & & \downarrow{\scriptstyle \beta(i \to j)} \\ \bullet & \xrightarrow{x_i} & \beta(i) \end{array}$$

[3] Here, $\mathsf{Set}(X, \beta) : I^{\mathrm{op}} \to \mathsf{Set}$ is a functor with $i \mapsto \mathsf{Set}(X, \beta(i))$.

Proof. For a map $f : X \to \varprojlim_I \beta$, which is defined by $f(x) = \{x_i\}_{i\in I} \in \varprojlim_I \beta$, consider the set of maps $\{f_i\}_{i\in I}$ such that $f_i : X \to \beta(i)$ with $f_i(x) = x_i$. Due to the property of $\{x_i\}$, $\beta(s)f_j(x) = f_i(x)$, hence the collection $\{f_i\}_{i\in I}$ is in the following set.

$$\varprojlim_I \mathsf{Set}(X,\beta) \simeq \left\{ \{f_i\}_{i\in I} \in \prod_{i\in I} \mathsf{Set}(X,\beta(i)) : \beta(s)f_j = f_i, \forall s \in I(i,j) \right\} \tag{4.4}$$

□

Proposition 4.1.5. *Let $\varphi : J \to I$ and $\beta : I^{op} \to$ Set be functors. Then there is a following natural morphism.*

$$\overline{\varphi} : \varprojlim_I \beta \to \varprojlim_J (\beta \circ \varphi^{op}) \tag{4.5}$$

Proof. Notice the followings.

$$\varprojlim_I \beta = \mathrm{Nat}(\{\bullet\},\beta)$$
$$\varprojlim_J \beta \circ \varphi^{\mathrm{op}} = \mathrm{Nat}(\{\bullet\},\beta\circ\varphi^{\mathrm{op}}) \tag{4.6}$$

Now we may define an isomorphism $\alpha : \{\bullet\} \Rightarrow \beta$ and $\overline{\varphi}\alpha : \{\bullet\} \Rightarrow \beta\circ\varphi^{\mathrm{op}}$ as $\overline{\varphi}(\alpha)_j = \alpha_{\varphi(j)}$, which is a natural morphism. □

4.2 *Limits on General Categories*

Definition 4.2.1. Let C be a category and $c \in$ C be an object.

1. Let $\alpha : I \to$ C be an inductive system and $\beta : I^{\mathrm{op}} \to$ C be a projective system. Then we define the functors $\varinjlim_I \alpha \in$ C $\to$ Set and $\varprojlim_I \beta \in$ C$^{\mathrm{op}} \to$ Set respectively, as followings.[4]

$$\varinjlim_I \alpha : c \mapsto \varprojlim_I \mathsf{C}(\alpha,c)$$
$$\varprojlim_I \beta : c \mapsto \varprojlim_I \mathsf{C}(c,\beta) \tag{4.7}$$

   If these functors are representable, then we write those representations with same notations[5] by abusing notations, calling them as **inductive limit** of α and **projective limit** of β, respectively.
2. For each inductive system $I \to$ C, suppose that $\varinjlim_I \alpha$ is representable. Then we say C **admits inductive limits indexed by** I.
   Dually, for each projective system $I^{\mathrm{op}} \to$ C, suppose that $\varprojlim_I \beta$ is representable. Then we say C **admits projective limits indexed by** I.
3. Let C admits inductive or projective limits indexed by all the finite or small categories. Then we say C admits **finite** or **small inductive** or **projective limits**.

[4] Here, as above, $\mathsf{C}(\alpha,c) : I \to$ Set is a functor with $i \mapsto \mathsf{C}(\alpha(i),c)$, and $\mathsf{C}(c,\beta) : I^{\mathrm{op}} \to$ Set is a functor with $i \mapsto \mathsf{C}*(c,\beta(i))$.

[5] which means, $\varinjlim_I \alpha$ and $\varprojlim_I \beta$.

| **Proposition 4.2.2.** | *There is a natural isomorphism between definitions of projective limit on Definition 4.1.2 and 4.2.1 if* $\mathsf{C} = \mathsf{Set}$.

Proof. This is just another way to read the Lemma 4.1.4. □

| **Proposition 4.2.3.** | *Let* $\alpha : I \to \mathsf{C}$ *be an inductive system. Then we have a following natural isomorphism.*[6]

$$\varinjlim \alpha \simeq \varprojlim \alpha^{op}. \tag{4.8}$$

[6] From now, we omit the subscript if the domain of inductive system is not important or well-known.

Proof. From the definition, we have $\varinjlim \alpha(c) = \varprojlim \mathsf{C}(\alpha, c)$ and $\varprojlim \alpha^{\mathrm{op}}(c) = \varprojlim \mathsf{C}^{\mathrm{op}}(c, \alpha) = \varprojlim \mathsf{C}(\alpha, c)$. Also, $\varinjlim \alpha(c \to d) = \varprojlim \mathsf{C}(\alpha, c \to d)$ and $\varprojlim \alpha^{\mathrm{op}}(c \to d)^{\mathrm{op}} = \varprojlim \mathsf{C}^{\mathrm{op}}(c \to d, \alpha) = \varprojlim \mathsf{C}(\alpha, c \to d)$. □

| **Proposition 4.2.4.** | *For* $\varphi : J \to I$, $\alpha : I \to \mathsf{C}$ *and* $\beta : I^{op} \to \mathsf{C}$ *are functors. Then we have following natural morphisms.*

$$\varinjlim(\alpha \circ \varphi) \to \varinjlim(\alpha) \tag{4.9}$$

$$\varprojlim \beta \to \varprojlim(\beta \circ \varphi^{op}) \tag{4.10}$$

Proof. This directly follows from the Proposition . □

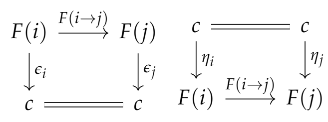
4.3 Limits as Universal Cones

| **Definition 4.3.1.** | The **diagonal functor** is a functor $\Delta : \mathsf{C} \to \mathsf{C}^J$ taking $f : c \to c'$ as $\Delta f : \Delta c \Rightarrow \Delta c'$, where $\Delta c : J \to \mathsf{C}$ is a functor with $\Delta c(i \to j) = 1_c$.

| **Definition 4.3.2.** | A **cone from** $F : J \to \mathsf{C}$ **to** $c \in \mathsf{C}$ is a natural transformation $\epsilon : F \Rightarrow \Delta c$.

Dually, a **cone from** $c \in \mathsf{C}$ **to** $F : J \to \mathsf{C}$ is a natural transformation $\eta : \Delta c \Rightarrow F$.[7]

[7]



| **Definition 4.3.3.** | A **colimit** of a functor $\alpha : J \to \mathsf{C}$ is a universal morphism $(\mathrm{colim}\alpha, \mu)$ from $\alpha \in \mathsf{C}^J$ to $\Delta : \mathsf{C} \to \mathsf{C}^J$. We call $\mathrm{colim}\alpha$ a **colimit object**, and $\mu : \alpha \Rightarrow \Delta(\mathrm{lim}\alpha)$ as a **colimit cone**.

Dually, a **limit** of a functor $\beta : J \to \mathsf{C}$ is a universal morphism $(\mathrm{lim}\beta, \nu)$ from $\Delta : \mathsf{C} \to \mathsf{C}^J$ to $\beta \in \mathsf{C}^J$. We call $\mathrm{lim}\beta$ a **limit object**, and $\nu : \Delta(\mathrm{lim}\beta) \Rightarrow \beta$ as a **limit cone**.

| **Lemma 4.3.4.** | *If* $\varinjlim \alpha$ *or* $\varprojlim \beta$ *are representable in* C, *then we get*

$$\mathsf{C}(\varinjlim \alpha, c) \simeq \varprojlim \mathsf{C}(\alpha, c) \tag{4.11}$$

and

$$\mathsf{C}(c, \varprojlim \beta) \simeq \varprojlim \mathsf{C}(c, \beta). \tag{4.12}$$

Proof. This naturally follows from the definition of representation functor. □

| THEOREM 4.3.5. | *Let $\alpha : J \to \mathsf{C}$ be an inductive system and $\beta : J^{op} \to \mathsf{C}$ a projective system. Then there is a following natural isomorphism between inductive limit and colimit if one of them exists, and dually, between projective limit and limit if one of them exists.*

$$\varinjlim \alpha \simeq \mathrm{colim}\alpha, \quad \varprojlim \beta \simeq \lim\beta \tag{4.13}$$

Proof. Limit case is the dual of colimit case. Because of the Proposition 3.2.2, there is a natural

$$\mathsf{C}(\mathrm{colim}\alpha, c) \simeq \mathrm{Nat}(\alpha, \Delta(c)) \tag{4.14}$$

for all object $c \in \mathsf{C}$. Also, by Lemma 4.3.4, we get

$$\mathsf{C}(\varinjlim \alpha, c) \simeq \varprojlim \mathsf{C}(\alpha, c) \simeq \mathrm{Nat}(\{\bullet\}, \mathsf{C}(\alpha, c)). \tag{4.15}$$

Now notice that $\mathrm{Nat}(\alpha, \Delta(c))$ and $\mathrm{Nat}(\{\bullet\}, \mathsf{C}(\alpha, c))$ are naturally isomorphic, with mapping from $\epsilon : \alpha \Rightarrow \Delta(c)$ to $\eta : \{\bullet\} \Rightarrow \mathsf{C}(\alpha, c)$ as $\eta_i(\bullet) = \epsilon_i$. Hence $\mathsf{C}(\mathrm{colim}, -) \simeq \mathsf{C}(\varinjlim \alpha, -)$, implying the desired bijection. □

| PROPOSITION 4.3.6. | *Let $F : I \to \mathsf{C}$ has its colimit $\varinjlim F$. Consider a cone $F \Rightarrow \Delta \varinjlim F$. Then for any cone $F \Rightarrow \Delta c$, there is a unique natural transformation $\Delta \varinjlim F \Rightarrow \Delta c$ factoring $F \Rightarrow \Delta c$.*

Dually, let $F : I \to \mathsf{C}$ has its limit $\varprojlim F$. Consider a cone $\Delta \varprojlim F \Rightarrow F$. Then for any cone $\Delta c \Rightarrow F$, there is a unique natural transformation $\Delta c \Rightarrow \Delta \varprojlim F$ factoring $\Delta c \Rightarrow F$.

Proof. Due to the Theorem 3.4.3, there is an initial object of $\int^{\mathsf{C}} \varinjlim F$. Theorem 4.3.5 then gives an initial object of $\mathrm{Nat}(F, \Delta c)$, which is a set of cone. The dual proof shows the dual statement. □

4.4 Special Limits and Colimits

| DEFINITION 4.4.1. | Let $F : I \to \mathsf{C}$ be a functor.

For a cone $\epsilon \in \mathsf{Nat}(F, \Delta(c))$, we call each $\epsilon_i : F(i) \to c$ a **leg** of a cone.

Dually, for a cone $\eta \in \mathsf{Nat}(\Delta(c), F)$, we call each $\eta_i : c \to F(i)$ a **leg** of a cone.

| DEFINITION 4.4.2. | Let $F : I \to \mathsf{C}$ be a functor, where I is a discrete category. Then we call $\varprojlim F$ a **product** of $\{F_i\}_{i \in I}$. We call each leg a **projection**. We often write as following.

$$\prod_{i \in I} F_i \coloneqq \varprojlim F, \quad p_j : \prod F_i \to F_j \tag{4.16}$$

If I is a discrete category with two objects 1 and 2, then we write

$$F_1 \times F_2 := \prod_{i \in I} F_i. \tag{4.17}$$

| Definition 4.4.3. | Let $F : I \to \mathsf{C}$ be a functor, with I is a discrete category. Then we call $\varinjlim F$ a **coproduct** of $\{F_i\}_{i \in I}$. We call each leg an **injection**. We often write as following.

$$\coprod_{i \in I} F_i := \varinjlim F, \quad i_j : F_j \to \coprod F_i \tag{4.18}$$

If I is a discrete category with two objects 1 and 2, then we write

$$F_1 \sqcup F_2 := \coprod_{i \in I} F_i. \tag{4.19}$$

| Definition 4.4.4. | Let $F : I \to \mathsf{C}$ be a functor, with I be a category with two objects $1, 2$ and two non-identity morphisms f, g from $1 \to 2$. Then we call $\varprojlim F$ an **equalizer** of $F(f)$ and $F(g)$.

| Definition 4.4.5. | Let $F : I \to \mathsf{C}$ be a functor, with I be a category with two objects $1, 2$ and two non-identity morphisms f, g from $1 \to 2$. Then we call $\varinjlim F$ a **coequalizer** of $F(f)$ and $F(g)$.

| Definition 4.4.6. | Let $F : I \to \mathsf{C}$ be a functor, with I be a category with three objects $1, 2, 3$ and two non-identity morphisms $f : 1 \to 2$, $g : 3 \to 2$. Then we call $\varprojlim F$ a **pullback** of $F(f), F(g)$.

| Definition 4.4.7. | Let $F : I \to \mathsf{C}$ be a functor, with I be a category with three objects $1, 2, 3$ and two non-identity morphisms $f : 2 \to 1$, $g : 2 \to 3$. Then we call $\varinjlim F$ a **pushout** of $F(f), F(g)$.

| Definition 4.4.8. | Let $F : \omega^{\mathrm{op}} \to \mathsf{C}$ be a functor where ω is a poset category on $\mathbb{N}$. Then we call $\varprojlim F$ an **inverse limit** of $\{F_i\}_{i \in \mathbb{N}}$.

| Definition 4.4.9. | Let $F : \omega \to \mathsf{C}$ be a functor. Then we call $\varinjlim F$ a **direct limit** of $\{F_i\}_{i \in \mathbb{N}}$.

4.5 Complete Category and Cocomplete Category

| Definition 4.5.1. | A category C is called a **complete category** if, for all small categories I and functors $F : I \to \mathsf{C}$, $\varprojlim F \in \mathsf{C}$.

Dually, a category C is called a **cocomplete category** if, for all small categories I and functors $F : I \to \mathsf{C}$, $\varprojlim F \in \mathsf{C}$.

Proposition 4.5.2. *The category* Set *is complete and cocomplete.*

Proof. This directly follows from the Definition 4.2.1. □

Definition 4.5.3. Let $U : \mathsf{C} \to \mathsf{X}$ be a functor. We call U **creates limits for a functor** $F : I \to \mathsf{C}$ if:

1. For every limiting cone $\epsilon : \Delta x \Rightarrow UF$, there is an object $c \in \mathsf{C}$ with $U(c) = x$, and a cone $\eta : \Delta c \Rightarrow F$ with $U\eta = \epsilon$;
2. This cone $\eta : \Delta c \Rightarrow F$ is a limiting cone.

Proposition 4.5.4. *Let* $U : \mathsf{Group} \to \mathsf{Set}$ *be the forgetful functor. Then it creates (co)limits.*

Proof. Let $F : I \to \mathsf{Group}$ be a functor. Consider two cones $\epsilon, \eta \in \mathrm{Nat}(\{\bullet\}, \mathsf{C}(UF, c))$ for some object $c \in \mathsf{C}$. Then we may define $(\epsilon \cdot \eta)_i := \epsilon_i \cdot \eta_i$ and $\left(\epsilon^{-1}\right)_i := \epsilon_i^{-1}$. Hence $\mathrm{Nat}(\{\bullet\}, \mathsf{C}(UF, c))$ has a group structure, and this group structure is unique.

Let G be a group with cone $\tau : \Delta G \Rightarrow F$ where $\tau_i LG \to F_i$. Then $U\tau : UG \Rightarrow UF$ is a cone, Thus by universality we have $U\tau_i = UG \Rightarrow UF$ is a cone in Set, hence there is a unique morphism $h : UG \to L$ Now,

$$h(g_1 g_2)_j = \lambda_j(g_1)\lambda_j(g_2) = (hg_1)_j(hg_2)_j = ((hg_1)(hg_2)) \qquad (4.20)$$

Hence h is a group homomorphism, showing that the limit is indeed in Grp.

The colimit case is the dual of limit case. □

Corollary 4.5.5. *A category* Group *is complete and cocomplete.*

Proof. This is the direct corollary. □

Proposition 4.5.6. *If a category* C *allows all equalizers and all products, then* C *is complete.*

Dually, if a category C *allows all coequalizers and all coproducts, then* C *is cocomplete.*

Proof. We only show the complete case here. Let $F : I \to \mathsf{C}$ be a functor. Then because of the property of product, there are two morphisms $f, g : \prod_i F_i \to \prod_{i\to j} F_j$, satisfying $p_{i\to j} f = p_j$ and $p_{i\to j} g = F(i \to j) p_i$.[8] Now consider an equalizer $e : c \to \prod_i F_i$. Define $\mu_i := p_i e : c \to F_i$. Then due to the product and equalizer property, $F(i \to j)\mu_i = \mu_j$, hence $\mu : \Delta c \Rightarrow F$ is a cone.

Choose another cone $\tau : \Delta d \Rightarrow F$. Then each morphisms $\tau_i : d \to F_i$ defines a unique map $h : d \to \prod_i F_i$ due to the product property, and $fh = gh$ due to the cone property. Hence h factors uniquely through e, implying τ factors uniquely through the cone μ. Thus μ is a limit cone. □

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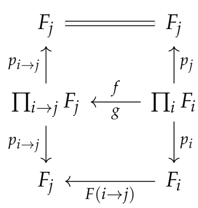

4.6
Continuous Functor

Definition 4.6.1. We call a functor $H : \mathsf{C} \to \mathsf{D}$ **continuous** if, for every functor $F : I \to \mathsf{C}$, $\varprojlim HF = H \varprojlim F$.

Dually, we call a functor $H : \mathsf{C} \to \mathsf{D}$ **cocontinuous** if, for every functor $F : I \to \mathsf{C}$, $\varinjlim HF = H \varinjlim F$.

Proposition 4.6.2. *A hom-functor* $\mathsf{C}(c, -)$ *is continuous.*

Proof. This directly follows from the Definition 4.2.1. □

Proposition 4.6.3. *Let* $U : \mathsf{C} \to \mathsf{X}$ *be a functor which creates limits for every functors* $F : I \to \mathsf{C}$ *with a limit of* $UF : I \to \mathsf{X}$*. Then* U *is continuous.*

Proof. Let $\epsilon : \Delta c \Rightarrow F$ and $\eta : \Delta x \Rightarrow UF$ be limiting cones. Because U creates limits, there is a unique limiting cone $\sigma : \Delta d \Rightarrow F$ with $U\sigma = \eta$. Since limits are unique up to isomorphism, we have an isomorphism $f : d \simeq c$ with $\epsilon f = \sigma$. Therefore $U(f) : U(d) = x \simeq U(c)$ and $(U\epsilon)\,(U(f)) = U\sigma = \eta$, hence $U\epsilon : \Delta U(c) \Rightarrow UF$ is a limiting cone. □

4.7
Limit as a Functor

Lemma 4.7.1. *Let* F, G *be a functor and* H *be a map between morphisms. If* $F = GH$ *and* G *is faithful, then* H *is a functor.*

Proof. We need to show that $H(gf) = H(g)H(f)$. Now, $GH(gf) = G(H(gf))$ and $F(gf) = F(g)F(f) = GH(g)GH(f) = G(H(g)H(f))$, hence $G(H(gf)) = G(H(g)H(f))$. Because G is faithful, $H(gf) = H(g)H(f)$. □

Theorem 4.7.2. *If a category* C *is complete, then* $\varprojlim$ *and* $\varinjlim$ *are functor* $\mathsf{C}^J \to \mathsf{C}$*.*

Proof. Because the constant functor $\Delta : \mathsf{C} \to \mathsf{C}^J$ is faithful, by Lemma 4.7.1, it is enough to show that $\Delta \varprojlim$ and $\Delta \varinjlim$ are functors. Because $\Delta \varinjlim$ case can be shown by taking dual statement of $\Delta \varprojlim$ case, we may only prove $\Delta \varprojlim$ case.

Let $F, F' : J \to \mathsf{C}$ be functors with natural transformation $\beta : F \Rightarrow F'$. Due to the limit property, there are unique limiting cones $\mu : \Delta \varprojlim F \Rightarrow F$ and $\mu' : \Delta \varprojlim F' \Rightarrow F'$. Then due to the limit property, there is a natural transformation $\Delta(\varprojlim \beta)$ satisfying $\beta\mu = \mu' \Delta(\varprojlim \beta)$. Due to the uniqueness, if there is a natural transformation $\alpha : F' \Rightarrow F''$, then $\Delta(\varprojlim \alpha)\Delta(\varprojlim \beta) = \Delta(\varprojlim \alpha\beta)$, we get the desired result. □

Chapter 5

Adjoint

Ah! Oh, the... accusative! Accusative! Ah!
'Domum', sir! 'Ad domum'! Ah! Oooh! Ah!
— Brian, *Life of Brian*

5.1 Adjoints and Adjunctions

Theorem 5.1.1. *Consider the functor $F : \mathsf{C} \to \mathsf{D}$, which defines a functor $F_* : \mathsf{Set}^{\mathsf{D}^{op}} \to \mathsf{Set}^{\mathsf{C}^{op}}$ as $F_*(H)(c) = H(F(c))$ for $H \in \mathsf{Set}^{\mathsf{D}^{op}}$ and $c \in \operatorname{ob} \mathsf{C}$.*

Suppose that the functor $F_* \circ Y_{\mathsf{D}}(d)$ is representable for each object $d \in \mathsf{D}$. Then there is a functor $\mathsf{D} \to \mathsf{C}$ such that $F_* \circ Y_{\mathsf{D}} \simeq Y_{\mathsf{C}} \circ G$, which is unique up to unique isomorphism.

Proof. Because each $F_* \circ Y_{\mathsf{D}}(d)$ is representable, the object in the image of $F_* \circ Y_{\mathsf{D}}$ is in the image of Y_{C}. Hence, because Y_{C} is fully faithful, we may take the quasi-inverse functor $I : \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{C}$. Thus $G = I \circ F_* \circ Y_D$ is the only possible such functor. □

Definition 5.1.2. Let $L : \mathsf{C} \to \mathsf{D}$ and $R : \mathsf{D} \to \mathsf{C}$ be two functors. Suppose thath there is a following natural isomorphism between bifunctors.

$$\mathsf{D}(L(-), -) \simeq \mathsf{C}(-, R(-)) : \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set} \tag{5.1}$$

Then we call (L, R) a pair of **adjoint functors**, L a **left adjoint** to R, and R a **right adjoint** to L.

Let $f : L(c) \to d$ for some $c \in \mathsf{C}, d \in \mathsf{D}$. Then we write the image of f under the isomorphism above as $f^\dagger : c \to R(d)$, and call it **adjoint** of f.[1]

[1] Notice that $(f^\dagger)^\dagger = f$.

Proposition 5.1.3. *Let $L : \mathsf{C} \to \mathsf{D}$ and $R : \mathsf{D} \to \mathsf{C}$ be two functors.*

1. *If L admits a right adjoint functor, this adjoint is unique up to unique isomorphism.*

2. *If R admits a left adjoint functor, this adjoint is unique up to unique isomorphism.*
3. *A functor L admits a right adjoint if and only if the functor $\mathsf{D}(L(-), d)$ is representable for any $d \in \mathsf{D}$.*

Proof. The functor $L_* \circ Y_{\mathsf{D}}$ can be considered as a functor $\mathsf{D}(L(-), -)$, and $Y_{\mathsf{C}} \circ R$ as $\mathsf{C}(-, R(-))$. Hence this is the reformulation of Theorem 5.1.1. □

Proposition 5.1.4. *Let $L : \mathsf{C} \to \mathsf{D}$ and $R : \mathsf{D} \to \mathsf{C}$ be two functors, and let $\epsilon : 1_{\mathsf{C}} \Rightarrow RL$ and $\eta : LR \Rightarrow 1_{\mathsf{D}}$ be natural transformations*[2] *satisfying the following* ***triangle identities****.*[3]

$$\begin{aligned} 1_L &= (\eta \circ L) \circ (L \circ \epsilon) : L \to LRL \to L \\ 1_R &= (R \circ \eta) \circ (\epsilon \circ R) : R \to RLR \to R \end{aligned} \tag{5.2}$$

Then (L, R) is a pair of adjoint functors. We say (L, R, η, ϵ) an ***adjunction*** *and ϵ, η the* ***adjunction morphisms****.*

Proof. What we need to show is the following two morphisms are inverse to each other.[4]

$$\begin{aligned} &\mathsf{D}(L(c), d) \xrightarrow{R} \mathsf{C}(RL(c), R(d)) \xrightarrow{\epsilon_c} \mathsf{C}(c, R(d)) \\ &\mathsf{C}(c, R(d)) \xrightarrow{L} \mathsf{D}(L(c), LR(d)) \xrightarrow{\eta_d} \mathsf{D}(L(c), d) \end{aligned} \tag{5.3}$$

Let $g : L(c) \to d$. Then by the first morphism, we get $R(g)\epsilon_c$, and by the second morphism, we get $\eta_d LR(g) L(\epsilon_c)$. Now due to the naturality of η, $\eta_d LR(g) L(\epsilon_c) = g \eta_{L(c)} L(\epsilon_c)$. Finally, due to the Equation 5.2, $\eta_{L(c)} L(\epsilon_c) = 1_{L(c)}$, thus g becomes g. Similarly, $f : c \to R(d)$ becomes $\epsilon_c RL(f) R(\eta_d)$, and $\epsilon_c RL(f) R(\eta_d) = f \epsilon_{R(d)} R(\eta_d)$, and $R(\eta_d)\epsilon_R(d) = 1_{R(d)}$ gives f becomes f. □

Lemma 5.1.5. *Let $L : \mathsf{C} \to \mathsf{D}$ and $R : \mathsf{D} \to \mathsf{C}$ be two functors with isomorphisms $\mathsf{D}(L(c), d) \simeq \mathsf{C}(c, R(d))$ for all $c \in \mathsf{C}$ and $d \in \mathsf{D}$. Then these isomorphisms are natural if and only if $kf = g^\dagger L(h)$ implies $R(k) f^\dagger = gh$ for all $L(c) \xrightarrow{f} d \xrightarrow{k} d'$ and $c \xrightarrow{h} c' \xrightarrow{g} R(d')$, and vice versa.*[5]

Proof. Notice that the naturality is equivalent to condition that, for all $k : d \to d'$, every $f : L(c) \to d$ satisfies $R(k) f^\dagger = (kf)^\dagger$, and for all $h : c \to c'$, every $g : c' \to R(d')$ satisfies $gh = (g^\dagger L(h))^\dagger$. Hence, $kf = g^\dagger L(h)$ if and only if $R(k) f^\dagger = gh$. The converse also can be shown in the same way. □

Proposition 5.1.6. *Let $\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$ be adjoint functors. Then there are natural transformations $\epsilon : 1_{\mathsf{C}} \Rightarrow RL$ and $\eta : LR \Rightarrow 1_{\mathsf{D}}$, satisfying the Equation 5.2.*

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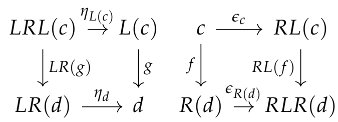

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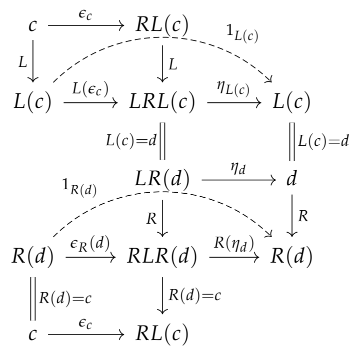

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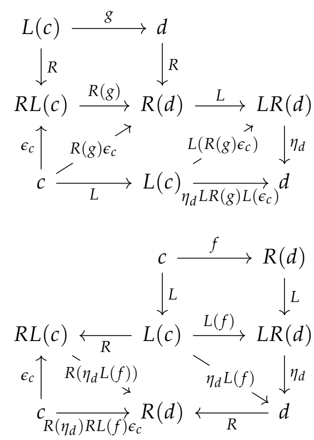

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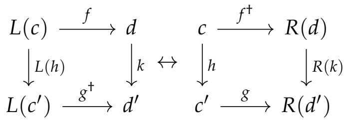


Proof. Define $\epsilon : 1_{\mathsf{C}} \Rightarrow RL$ as $\epsilon_c := 1^{\dagger}_{L(c)} : c \to RL(c)$. Because $L(f)1_{L(c)} = 1_{L(c')}L(f)$ for all $f : c \to c'$, by Lemma 5.1.5, we have $RL(f)1^{\dagger}_{L(c)} = 1^{\dagger}_{L(c')}f$, showing that ϵ is a natural transformation[6]. Similarly, we can define η. □

[6]
$$\begin{array}{ccc} L(c) & \xrightarrow{1_{L(c)}} & L(c) \\ \downarrow L(f) & & L(f) \downarrow \\ L(c') & \xrightarrow{1_{L(c')}} & L(c') \end{array} \quad \leftrightarrow \quad \begin{array}{ccc} c & \xrightarrow{\eta_c} & RL(c) \\ \downarrow f & & RL(f) \downarrow \\ c' & \xrightarrow{\eta_{c'}} & RL(c') \end{array}$$

Proposition 5.1.7 (Composition of adjoint functors). *Let $\mathsf{C}_{1,2,3}$ be categories with functors $\mathsf{C}_1 \underset{R_1}{\overset{L_1}{\rightleftarrows}} \mathsf{C}_2 \underset{R_2}{\overset{L_2}{\rightleftarrows}} \mathsf{C}_3$. If (L_1, R_1) and (L_2, R_2) are pairs of adjoint functors, then $(L_2 \circ L_1, R_1 \circ R_2)$ is a pair of adjoint functors.*

Proof. Take the objects $c_1 \in \mathsf{C}_1$ and $c_3 \in \mathsf{C}_3$. Then there are following functorial isomorphisms.

$$\begin{aligned} \mathsf{C}_3(L_2L_1(c_1), c_3) &\simeq \mathsf{C}_2(L_1(c_1), R_2(c_3)) \\ &\simeq \mathsf{C}_1(c_1, R_1R_2(c_3)) \end{aligned} \tag{5.4}$$

By definition this is a pair of adjoint functors. □

Proposition 5.1.8. *Let (L, R, η, ϵ) be an adjunction.*

1. *The functor L is fully faithful if and only if $\epsilon : 1_{\mathsf{C}} \Rightarrow RL$ is isomorphic.*
2. *The functor R is fully faithful if and only if $\eta : LR \Rightarrow 1_{\mathsf{D}}$ is isomorphic.*
3. *The followings are equivalent.*
   (a) *L is an equivalence of categories.*
   (b) *R is an equivalence of categories.*
   (c) *L and R are fully faithful.*

Proof. Because the second statement is dual of the first statement, and the third statement comes naturally from the third statement, we only need to show the first statement. Now L is fully faithful if and only if $\mathsf{C}(c, c') \simeq \mathsf{D}(L(c), L(c'))$, but $\mathsf{D}(L(c), L(c')) \simeq \mathsf{C}(c, RL(c'))$. Hence $\epsilon : 1_{\mathsf{C}} \Rightarrow RL$ is an isomorphism. □

5.2 Adjoints with Limits and Colimits

Lemma 5.2.1. *The functor $\varprojlim : \mathsf{C}^J \to \mathsf{C}$ is a right adjoint of $\Delta : \mathsf{C} \to \mathsf{C}^J$.*

Dually, the functor $\varinjlim : \mathsf{C}^J \to \mathsf{C}$ is a left adjoint of $\Delta : \mathsf{C} \to \mathsf{C}^J$.

Proof. This is just a repharsing of Lemma 4.3.4. □

Theorem 5.2.2. *Every right adjoint functors are continuous. Dually, every left adjoint functors are cocontinuous.*

Proof. Let (L, R, η, ϵ) be an adjunction between C and D. Then we have an a collection $(L^J, R^J, \eta^J, \epsilon^J)$ between C^J and D^J. Then the triangle identities still holds, so the collection is indeed an adjuntion. Now[7], for the left adjoints,

$$L^J \Delta = \Delta L \tag{5.5}$$

by definition, and their composition are again left adjoints, thus their right adjoints must commute:

$$\varprojlim R^J = R \varprojlim . \tag{5.6}$$

This shows that, for $F : J \to \mathsf{D}$, $\varprojlim R^J(F) = \varprojlim RF = R \varprojlim F$. □

[7] $$\begin{array}{ccc} \mathsf{C}^J & \underset{R^J}{\overset{L^J}{\rightleftarrows}} & \mathsf{D}^J \\ \Delta \upharpoonleft\downharpoonright \varprojlim & & \Delta \upharpoonleft\downharpoonright \varprojlim \\ \mathsf{C} & \underset{R}{\overset{L}{\rightleftarrows}} & \mathsf{D} \end{array}$$

5.3 Example: Tensor-Hom Adjunction

Proposition 5.3.1. *Let R, S be rings. Choose an (R, S)-bimodule X, and consider two functors*

$$- \otimes_R X : \mathsf{Mod}_R \to \mathsf{Mod}_S \tag{5.7}$$

$$\mathsf{Mod}_S(X, -) : \mathsf{Mod}_S \to \mathsf{Mod}_R \tag{5.8}$$

Then they are adjoint pairs: that is, there is a following natural isomorphism for all (A, R) bimodule Y and (B, S) bimodule Z with rings A, B.

$$\mathsf{Mod}_S(Y \otimes_R X, Z) \simeq \mathsf{Mod}_R(Y, \mathsf{Mod}_S(X, Z)) \tag{5.9}$$

Proof. It is enough to find out the adjunction morphisms. define $\epsilon : 1_{\mathsf{Mod}_S} \Rightarrow \mathsf{Mod}_S(X, - \otimes_R X)$ as

$$\epsilon_Y : Y \to \mathsf{Mod}_S(X, Y \otimes_R X), \quad \epsilon_Y(y)(x) = y \otimes x \tag{5.10}$$

and define $\eta : \mathsf{Mod}_R(X, -) \otimes_R X \Rightarrow 1_{\mathsf{Mod}_R}$ as

$$\eta : \mathsf{Mod}_R(X, Z) \otimes_R X \to Z, \quad \eta_Z(\phi \otimes x) = \phi(x). \tag{5.11}$$

Now we need to show the equation 5.2 holds. Because

$$\begin{aligned} &(\eta \circ - \otimes_R X) \circ (- \otimes_R X \circ \epsilon)_Y \\ &: Y \otimes_R X \to \mathsf{Mod}_S(X, Y \otimes_R X) \otimes_R X \to Y \otimes_R X \end{aligned} \tag{5.12}$$

takes

$$y \otimes x \mapsto \epsilon_Y(y)(-) \otimes x \mapsto \epsilon_Y(y)(x) = y \otimes x \tag{5.13}$$

and

$$\begin{aligned} &(\mathsf{Mod}_S(X, -) \circ \eta) \circ (\epsilon \circ \mathsf{Mod}_S(X, -))_Z \\ &: \mathsf{Mod}_S(X, Z) \to \mathsf{Mod}_S(X, \mathsf{Mod}_S(X, Z) \otimes_R X) \to \mathsf{Mod}_S(X, Z) \end{aligned} \tag{5.14}$$

takes

$$\phi(-) \mapsto \epsilon_{\mathsf{Mod}_S(X,Z)}(\phi)(-) = \phi \otimes - \mapsto \phi(-) \tag{5.15}$$

showing the desired result.

□

Corollary 5.3.2. *The Hom functor* $\mathsf{Mod}_S(X, -)$ *is continuous, and the tensor product functor* $- \otimes_R X$ *is cocontinuous.*

Proof. This directly follows from the Theorem 5.2.2 and Proposition 5.3.1. □

5.4 Example: Adjoint for Preorders

Theorem 5.4.1. *Let* P, Q *be two preorder*[8] *categories with order-preserving functors* $L : P \to Q^{op}$ *and* $R : Q^{op} \to P$*. Then* (L, R) *is an adjoint pair if and only if*

$$(L(p) \geq q) \Leftrightarrow (p \leq R(q)). \tag{5.16}$$

If so, we call L and R a ***Galois connection****. Thus,*

$$L(p) \geq LRL(p) \geq L(p), \quad R(q) \leq RLR(q) \leq R(q). \tag{5.17}$$

[8] A **preorder set** is a set with binary relation $\leq$ which is reflexive and transitive.

Proof. This directly follows from the definition of adjoint pair, and its triangular equalities. □

Example 5.4.2. Let G be a group acting on a set X. Take $P := \mathcal{P}(X)$ and $Q := \mathcal{P}(G)$. Define $L(S) := \{g : s \in S \Rightarrow gs = s\}$ and $R(H) := \{x : h \in H \Rightarrow hx = x\}$. Then we get

$$L(S) \geq H \Leftrightarrow hs = s \forall s \in S, h \in H \Leftrightarrow S \leq R(H). \tag{5.18}$$

Therefore L and R is a Galois connection.

2021.01.30.

Chapter 6

Abelian Category

And the Lord said unto Cain, Where is Abel thy brother?
And he said, I know not: Am I my brother's keeper?
— Genesis 4:9, *King James Version*

6.1 Pre-Additive Category

Definition 6.1.1. Let C be a category. Then we call C a **pre-additive category** if the set $\hom(c, d)$ is endowed with an abelian group structure[1] and the composition map is bilinear[2].

[1] Usually we write the binary operator as $+$, and call it **addition**.
[2] This is, $(f+g)\circ(h+k) = f\circ h + g\circ h + f\circ k + g\circ k$.

Example 6.1.2. 1. For a ring R, the module category Mod_R is an additive category. Here, the addition is defined as $(f+g)(m) = f(m) + g(m)$.

2. For a ring R, consider the category $\mathsf{B}R$, which is a category with one object, R morphisms, with their multiplication as composition. Using the addition in R, it is an additive category.

Proposition 6.1.3. *Let C be a preadditive category. Denote $0_{cd} \in \hom(c, d)$ as the identity element. Then the collection of 0_{cd} for all objects $c, d \in \mathsf{C}$ gives a category with zero morphisms.*

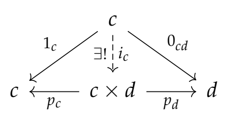
Proof. For any $f : b \to c$, $0_{cd}f = (0_{cd} + 0_{cd})f = 0_{cd}f + 0_{cd}f$, hence $0_{cd}f = 0_{bd}$. Similarly, $0_{ac} = f0_{ab}$. □

Lemma 6.1.4. *Let $c, d \in \mathsf{C}$ be objects in a pre-additive category C.*

1. *Let $c \times d \in \mathsf{C}$ with projections $p_c : c \times d \to c$ and $p_d : c \times d \to d$. Let $i_c : c \to c \times d$ be the[3] morphism defined by $p_c \circ i_c = 1_c$ and $p_d \circ i_c = 0_{cd}$. Similarly, let $i_d : d \to c \times d$ be the morphism defined by $p_c \circ i_d = 0_{dc}$ and $p_d \circ i_d = 1_d$. Then the following holds.*

$$i_c \circ p_c + i_d \circ p_d = 1_{c\times d} \tag{6.1}$$

[3] This morphism is unique due to the universal property of product.



2. *Conversely, let* $e \in \mathsf{C}$ *with* $\overline{p}_c : e \to c$, $\overline{p}_d : e \to d$, $\bar{i}_c : c \to e$, *and* $\bar{i}_d : d \to e$ *satisfying the above conditions. Then* e *is a product of* c *and* d *by* $(\overline{p}_c, \overline{p}_d)$ *and a coproduct by* $(\bar{i}_c, \bar{i}_d)$.

Proof.

1. Notice that,

$$p_c \circ (i_c \circ p_c + i_d \circ p_d) = 1_c \circ p_c + 0_{cd} \circ p_d = p_c \tag{6.2}$$

and similarly

$$p_d \circ (i_c \circ p_c + i_d \circ p_d) = 0_{cd} \circ p_c + 1_d \circ p_d = p_d \tag{6.3}$$

Therefore, due to the universal property of product, we get the desired result.

2. Consider the map $f : e \to c \times d$, which is uniquely induced by the maps $\overline{p}_c$ and $\overline{p}_d$. Also define $g = \bar{i}_c \circ p_c + \bar{i}_d \circ p_d$. [4] Now first we get the following.

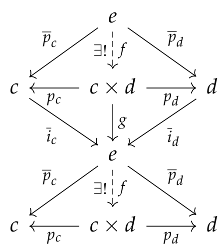
$$\begin{aligned} gf &= (\bar{i}_c p_c + \bar{i}_d p_d) f \\ &= \bar{i}_c \overline{p}_c + \bar{i}_d \overline{p}_d \\ &= 1_e \end{aligned} \tag{6.4}$$

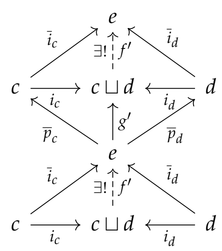
To show $fg = 1_{c\times d}$, notice the following.

$$\begin{aligned} \overline{p}_c g &= \overline{p}_c (\bar{i}_c p_c + \bar{i}_d p_d) \\ &= 1_c p_c + 0_{dc} p_d \\ &= p_c \end{aligned} \tag{6.5}$$

Similarly, $\overline{p}_d g = p_d$. Therefore, due to the universal property of product, $fg = 1_{c\times d}$.
Reversing all the arrows shows that e is a coproduct[5].

□

4 

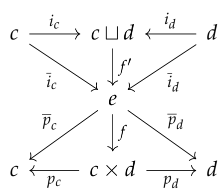
5 

| Corollary 6.1.5. | *Let* C *be a pre-additive category with objects* $c, d \in \mathsf{C}$. *Then* $c \sqcup d$ *exists if* $c \times d$ *exists, and there is the isomorphism* $r : c \sqcup d \to c \times d$ *satisfying the following.*

$$p_j \circ r \circ \bar{i}_k = \begin{cases} 1_j, & j = k \\ 0_{kj}, & j \neq k \end{cases} \tag{6.6}$$

Here, p_j *and* i_k *are projection and injection, respectively.*

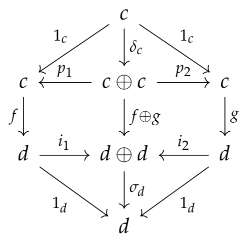
Proof. From Lemma 6.1.4, consider $f \circ f'$, which is isomorphism. Then $p_j \circ f \circ f' \circ i_k = \overline{p}_j \circ \bar{i}_k$, gives the desired result[6]. □

6 

| Definition 6.1.6. | Let $c, d \in \mathsf{C}$ be objects in pre-additive category. If $c \times d$ exists, then we write it $c \oplus d$, which is the **direct sum** of c and d.

Corollary 6.1.7. *Let* C *be a pre-additive category with objects* c, d *and morphisms* $f, g \in \hom(c, d)$*. Suppose that* $c \oplus c$ *and* $d \oplus d$ *exist with diagonal morphism* $\delta_c : c \to c \oplus c$ *and codiagonal morphism* $\sigma_d : d \oplus d \to d$*. Then* $f + g = \sigma_d \circ (f_1 \oplus f_2) \circ \delta_c$*.*

Proof. Denote $p_i : c \oplus c \to c$ be the projection maps and $i_j : d \to d \oplus d$ be the injection maps. Notice that $f \oplus g = i_1 f p_1 + i_2 f p_2$, therefore $\sigma_d \circ (f \oplus g) \circ \delta_c = 1_d \circ f \circ 1_c + 1_d \circ g \circ 1_c = f + g$.[7] □

[7]


Definition 6.1.8. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor of pre-additive categories. Then we say F is an **additive functor** if $F_{cd} : \mathsf{C}(c, d) \to \mathsf{D}(F(c), F(d))$ is a group homomorphism for any $c, d \in \mathsf{C}$.

6.2 *Additive Category*

Definition 6.2.1. A category C is **additive category** if:

1. C has a zero object 0, and thus zero morphisms $0_{cd} : c \to 0 \to d$;
2. For any objects $c, d \in \mathsf{C}$, $c \times d \in \mathsf{C}$ and $c \sqcup d \in \mathsf{C}$;
3. For any objects $c, d \in \mathsf{C}$, define the morphism $r : c \sqcup d \to c \times d$ satisfying the following.

$$p_j \circ r \circ \bar{i}_k = \begin{cases} 1_j, & j = k \\ 0_{kj}, & j \neq k \end{cases} \tag{6.7}$$

   Here, p_j and i_k are projection and injection, respectively. Then r is an isomorphism.
4. For any $c \in \mathsf{C}$, there is an endomorphism $f \in \hom(c, c)$ such that the composition

$$c \xrightarrow{\delta_c} c \times c \xrightarrow{(f, 1_c)} c \times c \xleftarrow{r} c \sqcup c \xrightarrow{\sigma_c} c \tag{6.8}$$

   is the zero morphism. Here, δ_c is the diagonal morphism, and δ_c is the codiagonal morphism.

Proposition 6.2.2. *A pre-additive category with finite products is additive.*

Proof. By Corollary 6.1.5 and 6.1.7, with $a = -1_c$, we get the desired result. □

Example 6.2.3. Let R be a ring. Then the module category Mod_R and finitely generated module category $\mathsf{Mod}_R^{\mathsf{f}}$ are additive category.

Definition 6.2.4. Let C be an additive category. Then a **chain complex** $c^\bullet$ **in an additive category** C is a sequence of objects $\{c^j\}_{j \in \mathbb{Z}}$ and morphisms $d_c^j : c^j \to c^{j+1}$ such that $d_c^j d_c^{j-1} = 0_{c^{j-1} c^{j+1}}$ for all $j \in \mathbb{Z}$.

6.3 Subobjects and Elements

Definition 6.3.1. Let C be a category. For two monics $u : a \to c$ and $v : b \to c$, there is a partial order $u \leq v$ when $u = vw$ for some (monic) w, with equivalence relation $u \equiv v$. We call each equivalence class of these monics a **subobject**.

Dually, for two epis $r : a \to b$ and $s : a \to c$, there is a partial order $r \leq s$ when $r = qs$ for some (epi) q, with equivalence relation $r \equiv s$. We call each equivalence class of these epis a **quotient object**.

Proposition 6.3.2. *Let two subobjects u, v are equivalent. Then there is an isomorphism f such that $u = vf$.*

Proof. By definition, $u = vf$ and $v = ug$. Thus $u = ugf$ and $v = vfg$, which implies $gf = 1$ and $fg = 1$ because u, v are monics. Therefore f, g are isomorphisms. □

Lemma 6.3.3. *Pullbacks of monics are monics. Dually, pushforwards of epis are epis.*

Proof. Let the pullback of f, g as f', g', respectively. Suppose that f is monic[8]. Consider a parallel pair h, k satisfying $f'h = f'k$. Then $gf'h = gf'k$, thus by commutativity, $fg'h = fg'k$. Since f is monic, $g'h = g'k$. Thus, by the universality of pullback, $h = k$. □

[8] The case g is monic can be shown in the same way.

Definition 6.3.4. Let C be a complete category. Then for two subobjects u, v, a subobject w given by the pullback of u, v is called the **intersection**.

Definition 6.3.5. Let $x : b \to c$ for $b, c \in \mathsf{C}$. Then we write $x \in c$, calling x an **element** of a. We write $x \equiv y$ for two $x, y \in c$ if there are epis u, v with $xu = yv$.

6.4 Abelian Category

Definition 6.4.1. Let C be an additive category. We say C is an **abelian category** if:

1. every morphism in C admits a kernel and a cokernel;
2. every monomorphism is a kernel, and every epimorphism is a cokernel.

Lemma 6.4.2. *Let C be an abelian category with morphism f. Then the followings hold.*

$$\operatorname{Ker}\operatorname{coKer}\operatorname{Ker} f = \operatorname{Ker} f, \quad \operatorname{coKer}\operatorname{Ker}\operatorname{coKer} f = \operatorname{coKer} f \tag{6.9}$$

Proof. Let P_c be the set of morphisms with codomain c, and Q^c be the set of morphisms with domain c. Then there is a preorder on P_c saying $g \leq f$ if $g = fg'$ for some g', and a preorder on Q^c saying $u \geq v$ if $v = v'u$ for some v'.

Now, due to the universal properties of kernel and cokernel,

$$f \leq \operatorname{Ker} u \Leftrightarrow uf = 0 \Leftrightarrow \operatorname{coKer} f \geq u. \tag{6.10}$$

Thus $(\operatorname{Ker}, \operatorname{coKer})$ is a Galois connection, and from the triangular identities, we get the followings.

$$\operatorname{Ker} f \geq \operatorname{Ker} \operatorname{coKer} \operatorname{Ker} f \geq \operatorname{Ker} f, \tag{6.11}$$

$$\operatorname{coKer} f \leq \operatorname{coKer} \operatorname{Ker} \operatorname{coKer} f \leq \operatorname{coKer} f \tag{6.12}$$

By the Proposition 6.3.2, considering each kernels and cokernels as objects, we get the desired result. □

Proposition 6.4.3. *Let* C *be an abelian category. Then* m *is monic if and only if* $\operatorname{Ker} \operatorname{coKer} m = m$. *Dually,* e *is epi if and only if* $\operatorname{coKer} \operatorname{Ker} e = e$.

Proof. Let m be a monomorphism. Because m is a morphism of abelian cataegory, $m = \operatorname{Ker} f$ for some f. Thus by Lemma 6.4.2, $\operatorname{Ker} \operatorname{coKer} \operatorname{Ker} f = \operatorname{Ker} f$ implies $\operatorname{Ker} \operatorname{coKer} m = m$. Conversely, let $\operatorname{Ker} \operatorname{coKer} m = m$. Because m is a kernel, m is monic. □

Lemma 6.4.4. *The pullback of epi in an abelian category is epi. Dually, the pushout of monic in an abelian category is monic.*

Proof. Let the pullback of epis $f : b \to c, g : d \to c$ be f', g'. Now consider the following sequence.

$$s \xrightarrow{m} b \oplus d \xrightarrow{fp_1 - gp_2} c \tag{6.13}$$

Here m is a kernel of $fp_1 - gp_2$, and p_1, p_2 are projections of $b \oplus d$. Then we may let $f' = p_2 m$ and $g' = p_1 m$.

Suppose that $h(fp_1 - gp_2) = 0$. Then using the injection i_1, we get

$$0 = h(fp_1 - gp_2)i_1 = hf \tag{6.14}$$

but since f is epi, $h = 0$. Thus $fp_1 - gp_2$ is an epi, and thus by Proposition 6.4.3, $\operatorname{coKer} m = fp_1 - gp_2$. Now let $uf' = 0$ for some u. Then $up_2 m = 0$, and thus up_2 factors through $\operatorname{coKer} m = fp_1 - gp_2$, as $up_2 = u'(fp_1 - gp_2)$. Thus,

$$0 = up_2 i_1 = u'(fp_1 - gp_2)i_1 = u'f \tag{6.15}$$

but since f is epi, $u' = 0$. □

| **Proposition 6.4.5.** | *Let C be an abelian category and f is a morphism in C. Then there is a factorization $f = me$ with monic m and epic e. Furthermore, if there is another factorization $f' = m'e'$ by some monic m' and epic e', and $f'g = hf$, then there is a unique k satisfying $e'g = ke$ and $m'k = hm$.*[9] *Hence, a factorization of f is unique up to isomorphism.*

For a factorization $f = me$, we may take

$$m = \operatorname{Im} f := \operatorname{Ker}(\operatorname{coKer} f), \quad e = \operatorname{coIm} f := \operatorname{coKer}(\operatorname{Ker} f). \tag{6.16}$$

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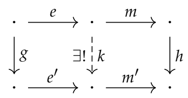

Proof. $f(\operatorname{Ker} f) = 0$ implies $\exists! u$ with $ue = f$, and $(\operatorname{coKer} f)f = 0$ implies $\exists! v$ with $mv = f$. Now $mv(\operatorname{Ker} f) = 0$ implies $v(\operatorname{Ker} f) = 0$, which implies $\exists! g$ with $ge = v$, and $(\operatorname{coKer} f)ue = 0$ implies $(\operatorname{coKer} f)u = 0$, which implies $\exists! h$ with $mh = u$. Hence in total, $mge = f = mhe$. Now $m(g - h)e = 0$ implies $g = h$.

To show that g is monic, suppose that $ga = 0$. Then we may take a pullback of e and a, satisfying $ei = aj$. Since $fi = mgei = mgaj = 0$, i factors as $i = (\operatorname{Ker} f)t$, thus $aj = e(\operatorname{Ker} f)t$. But since $e(\operatorname{Ker} f) = 0$, $aj = 0$, and since j is the pullback of epimorphism e, j is epi, therefore $a = 0$.

To show that h is epi, suppose that $bh = 0$. Then we may take a pushout of m and b, satisfying $km = lb$. Since $kf = kmhe = lbhe = 0$, k factors as $k = s(\operatorname{coKer} f)$, thus $lb = s(\operatorname{coKer} f)m$. But since $(\operatorname{coKer} f)m = 0$, $lb = 0$, and since l is the pushout of monomorphism m, l is monic, therefore $b = 0$.[10] □

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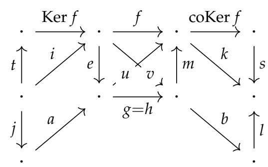

6.5 *Exact Sequence*

| **Definition 6.5.1.** | Consider following pair of morphisms in an Abelian category.

$$\cdot \xrightarrow{f} b \xrightarrow{g} \cdot \tag{6.17}$$

Then this sequence is **exact** when $\operatorname{Im} f \equiv \operatorname{Ker} g$, where the equivalence comes from the subobjects of b.

If the following diagram is exact everywhere,

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0 \tag{6.18}$$

where 0 is the zero object, we call it a **short exact sequence**.

If the following diagram is exact everywhere,

$$a \xrightarrow{f} b \xrightarrow{g} c \to 0 \tag{6.19}$$

we call it a **short right exact sequence**.

If the following diagram is exact everywhere,

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \tag{6.20}$$

we call it a **short left exact sequence**.

Proposition 6.5.2. $\operatorname{Im} f \leq \operatorname{Ker} g$ *if and only if* $gf = 0$. *Also,* $\operatorname{Im} f \geq \operatorname{Ker} g$ *if and only if every* k *with* $gk = 0$ *factors as* $k = mk'$ *for the monic-epi factorization* $f = me$.[11]

[11] In other words, (f, g) is exact if and only if the composition gf is a zero map and every element killed by g is in the image of f.

Proof. □

Definition 6.5.3. Let C be a category with finite limits. Then a functor $F : \mathsf{C} \to \mathsf{D}$ is **left exact** if it commutes with finite limits.

Dually, let C be a category with finite colimits. Then a functor $F : \mathsf{C} \to \mathsf{D}$ is **right exact** if it commutes with finite colimits.

If a functor F is both left and right exact, we call it **exact functor**.

Proposition 6.5.4. *Let* C *be an abelian category and* F *is an additive functor. Then the followings are equivalent.*

1. *F is left exact.*
2. *Whenever* $0 \to A \to B \to C \to 0$ *is exact in* C, $0 \to F(A) \to F(B) \to F(C)$ *is exact.*
3. *Whenever* $0 \to A \to B \to C$ *is exact in* C, $0 \to F(A) \to F(B) \to F(C)$ *is exact.*

Also the followings are equivalent.

1. *F is right exact.*
2. *Whenever* $0 \to A \to B \to C \to 0$ *is exact in* C, $F(A) \to F(B) \to F(C) \to 0$ *is exact.*
3. *Whenever* $A \to B \to C \to 0$ *is exact in* C, $F(A) \to F(B) \to F(C) \to 0$ *is exact.*

Proof. □

Definition 6.5.5. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in an abelian category C. If there is $h : C \to B$ and $k : B \to A$ such that $1_B = fk + hg$, then we call the sequence **splits**.

Proposition 6.5.6. *For a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in an abelian category* C, *the followings are equivalent.*

1. *The short exact sequence splits.*
2. *There is* $h : C \to B$ *such that* $gh = 1_C$.
3. *There is* $k : B \to A$ *such that* $kf = 1_A$.
4. *There are* $(k, g) : B \to A \oplus C$ *and* $(f, h) : A \oplus C \to B$ *which are isomorphisms to each other.*
5. *f is a split monomorphism.*
6. *g is a split epimorphism.*

Proof.

(1 ⇒ 2). Let $1_B = fk + hg$. Because $gf = 0$, $g = gfk + ghg = ghg$, thus $(gh - 1_C)g = 0$. Since g is epi, $gh = 1_C$.

(2 ⇒ 1). Let $gh = 1_C$. Because $g = ghg$, we have $g(1_B - hg) = 0$. Thus $1_B - hg$ can be factorized by $k : B \to A$, that is, $1_B - hg = fk$.
(2 ⇔ 3). This proof is basically the dual of above.
(1 ⇔ 4). This follows naturally from aboves.
(1 ⇔ 5, 6). This follows naturally from the Theorem 1.3.12.

□

| **Definition 6.5.7.** | An abelian category is called **semisimple** if all short exact sequences split.

6.6 Diagram Chasing

| **Lemma 6.6.1.** | *The relation $x \equiv y$ of two elements in Abelian category are transitive, hence an equivalence relation. That is, $x \equiv y$ and $y \equiv z$ implies $x \equiv z$.*

Proof. Let $x \equiv y$ and $y \equiv z$, that is, there are epis u, v, w, r with $xu = yv$ and $yw = zr$.[12] Then we may take a pullback of w and r, say w' and r'. By Lemma 6.4.4, w', r' are epis, thus uw' and rv' are epis, saying $x \equiv z$. □

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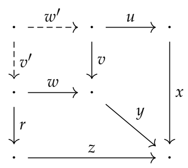

| **Theorem 6.6.2 (Rules of diagram chasing).** | *In abelian category, the followings hold.*

1. *(Monomorphism 1) $f : a \to b$ is monic if and only if $fx \equiv 0$ implies $x \equiv 0$ for all $x \in a$.*
2. *(Monomorphism 2) $f : a \to b$ is monic if and only if $fx \equiv fx'$ implies $x \equiv x'$ for all $x, x' \in a$.*
3. *(Epimorphism) $g : b \to c$ is epi if and only if there is a $y \in b$ with $gy \equiv z$ for all $z \in c$.*
4. *(Zero morphism) $h : r \to s$ is the zero arrow if and only if $hx \equiv 0$ for all $x \in r$.*
5. *(Exact sequence) A sequence $a \xrightarrow{f} b \xrightarrow{g} c$ is exact at b if and only if $gf = 0$ and for every $y \in b$ with $gy \equiv 0$ there is $x \in a$ with $fx \equiv y$.*
6. *(Substraction) Let $g : b \to c$ and $x, y \in b$ with $gx \equiv gy$. Then there is an element $z \in b$ with $gz \equiv 0$. Furthermore, any $f : b \to d$ with $fx \equiv 0$ has $fy \equiv fz$ and any $h : b \to a \equiv 0$ has $hx \equiv -hz$.*

6.7 Snake, Five, and Nine lemma

| **Lemma 6.7.1 (Snake lemma).** |

Lemma 6.7.2 (Five lemma).

Lemma 6.7.3 (Nine lemma).

6.8 Injective and Projective Objects

Definition 6.8.1. Let C be a category. A set S of objects of the category C **generates** C if for any parallel pair $g, h : c \to d$, $g \neq h$ implies that there is an object $s \in S$ and a morphism $f : s \to c$ such that $gf \neq hf$.

Dually, a set Q of objects of the category C **cogenerates** C if for any parallel pair $g, h : c \to d$, $g \neq h$ implies that there is an object $q \in Q$ and a morphism $f : d \to q$ such that $fg \neq fh$.

Proposition 6.8.2. *Let S be a subset of objects in* C*. Then the followings are equivalent.*

1. *S generates* C.
2. *For any $c \in$ C, there is $s \in S$ with an epimorphism $s \to c$.*

Dually, the followings are equivalent.

1. *Q cogenerates* C.
2. *For any $c \in$ C, there is $q \in Q$ with a monomorphism $c \to q$.*

Proof. (1 ⇒ 2). Suppose not, that is, for all morphism $f : s \to c$ with $s \in S$, f is not an epimorphism. This implies that there is $g, h : c \to d$ such that $gf = hf$ for all f but $g \neq h$, coontradiction.
(2 ⇒ 1). Let $g, h : c \to d$ with $g \neq h$. By hypothesis, there is $s \in S$ with an epimorphism $f : s \to c$. If $gf = hf$ then since f is epi $g = h$, contradiction, thus $gf \neq hf$. □

Definition 6.8.3. Let C be an abelian category.

1. An object I in C is **injective** if the functor $\mathsf{C}(-, I)$ is exact.
2. Dually, an object P in C is **projective** if the functor $\mathsf{C}(P, -)$ is exact.

Let $\{I\}$ be the set of injective objects of C, and $\{P\}$ be the set of projective objects of C.

1. We call C has **enough injectives** if $\{I\}$ cogenerates C.
2. We call C has **enough projectives** if $\{P\}$ generates C.

Proposition 6.8.4. *An object $I \in$ C is injective if and only if, for any subobject $f : X \to Y$ and a map $k : X \to I$, there is a map $h : Y \to I$ with $k = hf$.*[13]

Dually, an object $P \in$ C is projective if and only if, for any quotient object $q : X \to Y$ and a map $m : P \to Y$, there is a map $h : P \to X$ with $m = qh$.

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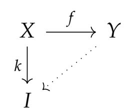


Proof. Consider an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{\mathrm{coKer}\, f} Z \to 0$. Then because $\mathsf{C}(-, I)$ is left exact, $0 \to \mathsf{C}(Z, I) \to \mathsf{C}(Y, I) \to \mathsf{C}(X, I)$ is exact. Thus $0 \to \mathsf{C}(Z, I) \to \mathsf{C}(Y, I) \to \mathsf{C}(X, I) \to 0$ is exact if and only if $\mathsf{C}(Y, I) \to \mathsf{C}(X, I)$ is surjective. □

2021.03.02.

Part II

Homological Algebra

Chapter 7

Chain complex and Homology

Chain Lightning deals 3 damage to any target. Then that player or that permanent's controller may pay RR. If the player does, they may copy this spell and may choose a new target for that copy.

— Chain Lightning, *Magic: the gathering*

7.1 Chain complex

DEFINITION 7.1.1. A **chain complex** $C_\bullet$ of an abelian category C is a family $\{C_n\}_{n\in\mathbb{Z}}$ of objects in C with maps $d_n : C_n \to C_{n-1}$, called **differentials**, satisfying $d_{n-1} \circ d_n = 0$. We write the chain complex as $C_\bullet$.

A **chain map** $f_\bullet$ between chain complexes $C_\bullet$ and $D_\bullet$ is a family $\{f_n\}_{n\in\mathbb{Z}}$ of morphisms in C which satisfies $f_{n-1}d_n^{\mathsf{C}} = d_n f_n$.

For two chain maps $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to E_\bullet$, the composition of chain maps $(gf)_\bullet$ is defined as $(gf)_\bullet = g_\bullet \circ f_\bullet$.

A **chain complex category** $\mathsf{Ch}(\mathsf{C})$, or simply Ch, is a category with chain complexes as objects and chain maps as morphisms.

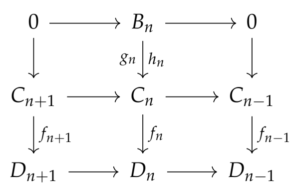
LEMMA 7.1.2. *Let $f_\bullet$ be a chain map. Then it is monic if and only if $\{f_n\}_{n\in\mathbb{Z}}$ is a set of monomorphisms.*

Dually, $f_\bullet$ is epi if and only if $\{f_n\}_{n\in\mathbb{Z}}$ is a set of epimorphisms.

Proof. Suppose that $f_\bullet$ is a chain map. For f_n, suppose that $g_n f_n = h_n f_n$ for some composable morphisms g_n, h_n. Now consider the chain complex, with everywhere zero morphisms and maps except g_n and h_n.[1] Because $f_\bullet$ is monic, $g_n = h_n$.

Suppose that $\{f_n\}_{n\in\mathbb{Z}}$ is a set of monomorphisms. Suppose that $g_\bullet, h_\bullet$ are composable chain maps with $gf_\bullet = hf_\bullet$. Because $gf_\bullet = g_\bullet f_\bullet = h_\bullet f_\bullet = hf_\bullet$, $g_\bullet = h_\bullet$. □

[1]



LEMMA 7.1.3. *Let $f_\bullet$ be a chain map. Then the kernel chain complex $(\mathrm{Ker}\, f)_\bullet := \mathrm{Ker}(f_\bullet)$ is a kernel of $f_\bullet$.*

Dually, the cokernel chain complex $(\mathrm{coKer}\, f)_\bullet := \mathrm{coKer}(f_\bullet)$ is a cokernel of $f_\bullet$.

Proof. First, due to the kernel property, there exists a kernel chain $\operatorname{Ker} f$.[2] To show that this is kernel, suppose that $gf_\bullet = 0$, that is, $g_\bullet f_\bullet = 0$. Due to the kernel property, this can be factorized by $\operatorname{Ker} f_\bullet$, which indeed gives a chain map because the kernel map is monic. □

[2]
$$\begin{array}{ccc} \operatorname{Ker} f_{n+1} & \longrightarrow & \operatorname{Ker} f_n \\ \downarrow & & \downarrow \\ C_{n+1} & \longrightarrow & C_n \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} \\ D_{n+1} & \longrightarrow & D_n \end{array}$$

| PROPOSITION 7.1.4. | *Let* C *be an abelian category. Then the category* $\mathsf{Ch}(\mathsf{C})$ *is an abelian category.*

Proof. By the obvious definition of addition, $(f+g)_\bullet = f_\bullet + g_\bullet$, $\mathsf{Ch}(\mathsf{C})$ is a pre-additive category. Because the finite product $\left(\oplus_i C^i\right)_\bullet$ of chain complexes always exists as $\oplus_i \left(C^i_\bullet\right)$, and the zero object exists, it is an additive category, by Proposition 6.2.2. Finally, by Lemma 7.1.2 and 7.1.3, we get the desired result. □

| DEFINITION 7.1.5. | Let $C_\bullet$ be a chain complex. For some integer p, we define a **translated chain complex** $C[p]_\bullet$ as a chain complex with

$$C[p]_n = C_{n+p} \tag{7.1}$$

with differentials $d_n^{C[p]} = (-1)^p d_{n+p}^C$.[3]

[3] The sign $(-1)^p$ is useful to simplify further notations.

| COROLLARY 7.1.6. | *Let* $C_\bullet$ *be a chain complex. Then* $H_{n-p}(C[p]) \simeq H_n(C)$.

Proof. This follows naturally from the definition of translated chain complex. □

7.2 Homology

| DEFINITION 7.2.1. | Let $C_\bullet$ be a chain complex with differential $d_\bullet$.

1. The kernel of d_n is called the module of n-**cycle**, often written as $Z_n(C)$ or simply Z_n.
2. The image of d_{n+1} is called the module of n-**boundary**, often written as $B_n(C)$ or simply B_n.

| PROPOSITION 7.2.2. | *There is a natural monomorphism* $B_n \to Z_n$.

Proof. We may factorize d_{n+1} by $m_{n+1}e_{n+1}$, with monic m_{n+1} and epi e_{n+1}. Because $d_n d_{n+1} = d_n m_{n+1} e_{n+1} = 0$, by epi property, $d_n m_{n+1}$, and thus m_{n+1} factorizes by $\operatorname{Ker} d_n$. Because m_{n+1} and $\operatorname{Ker} d_n$ are monic and the domain of m_{n+1} is isomorphic to $\operatorname{Im} d_{n+1}$, we get the desired monomorphism. □

| Definition 7.2.3. | Let $C_\bullet$ be a chain complex with natural monomorphisms $B_n \to Z_n$. Then the cokernel of this map, which is a quotient object, is called a **homology**, often written as $H_n(C)$ or simply H_n.

If $H_n(C) = 0$ for all $n \in \mathbb{Z}$, we call $C_\bullet$ **acyclic**.

| Proposition 7.2.4. | *For a chain map $f_\bullet : C_\bullet \to D_\bullet$, there are natural morphisms $B_n(C) \to B_n(D)$, $Z_n(C) \to Z_n(D)$, and thus $H_n(C) \to H_n(D)$. Therefore, $B_n, Z_n, H_n : \mathsf{Ch}(\mathsf{C}) \to \mathsf{C}$ are additive functors for all $n \in \mathbb{Z}$.*

Proof. Consider the decomposition of map $C_{n+1} \xrightarrow{d^C_{n+1}} C_n$ into $C_{n+1} \xrightarrow{\text{coKer}\,\text{Ker}\, d^C_{n+1}} B_n(C) \xrightarrow{m^C_n} Z_n(C) \xrightarrow{\text{Ker}\, d^C_n} C_n$, and similar on $D_\bullet$. Because $0 = \text{Ker}\, d^C_{n+1} \to C_{n+1} \to C_n \to D_n = \text{Ker}\, d^C_{n+1} \to C_{n+1} \xrightarrow{f_{n+1}} D_{n+1} \to B_n(D) \to Z_n(D) \to D_n$, and the last two maps are monic, $\text{Ker}\, d^C_{n+1} \to C_{n+1} \xrightarrow{f_{n+1}} D_{n+1} \to B_n(D)$ is a zero map. Thus $C_{n+1} \xrightarrow{f_{n+1}} D_{n+1} \to B_n(D)$ can be factorized by $\text{coKer}\,\text{Ker}\, d^C_{n+1}$, giving a map $B_n(C) \to B_n(D)$.

The kernel map $Z_n(C) \to Z_n(D)$, and cokernel map $H_n(C) \to H_n(D)$, are induced naturally by the kernel property. □

| Definition 7.2.5. | A chain map $f_\bullet$ is called a **quasi-isomorphism** if the maps $H_n(C) \to H_n(D)$ naturally induced by $f_\bullet$ are isomorphisms.

| Proposition 7.2.6. | *Let $C_\bullet$ be a chain complex. Then the followings are equivalent.*

1. *$C_\bullet$ is exact.*
2. *$C_\bullet$ is acyclic.*
3. *$C_\bullet$ is quasi-isomorphic to $0_\bullet$.*

Proof.

(1 ⇔ 2.) $Z_n \simeq B_n$ if and only if $H_n = 0$ for all $n \in \mathbb{Z}$.
(2 ⇔ 3.) $0_\bullet$ has zero homology modules.

□

7.3 Homology Long Exact Sequence

| Lemma 7.3.1. | *Let $C_\bullet$ be a chain complex. Then the followings are exact sequences.*

1. $0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0$
2. $0 \to Z_n(C) \to C_n \to B_{n-1}(C) \to 0$
3. $0 \to H_n(C) \to \text{coKer}(d_{n+1}) \to \text{Ker}(d_{n-1}) \to H_{n-1}(C) \to 0$

Proof.

1. Because $\operatorname{Ker}(Z_n \to H_n) = \operatorname{Ker}\operatorname{coKer}(B_n \to Z_n) = \operatorname{Im}(B_n \to Z_n)$, and $B_n \to Z_n$ is monic and $Z_n \to H_n$ is epi, the sequence is exact.
2. Because $\operatorname{Ker}(C_n \to B_{n-1}) = \operatorname{Ker} d_n = Z_n$, we get the desired result.
3. Because of the first statement, it is enough to show that $0 \to H_n(C) \to \operatorname{coKer}(d_{n+1}) \to B_{n-1} \to 0$ is exact. Because of the two above, we can build the following commutative diagram, where all the horizontal rows are exact and the first two vertical columns are exact. By the lemma 6.7.3, the last column is exact.

$$\begin{array}{ccccc} B_n & \longrightarrow & Z_n & \longrightarrow & H_n \\ \downarrow{\scriptstyle 1_{B_n}} & & \downarrow & & \downarrow \\ B_n & \longrightarrow & C_n & \longrightarrow & \operatorname{coKer} d_{n+1} \\ \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow \\ B_n & \xrightarrow{0} & B_{n-1} & \xrightarrow{1_{B_{n-1}}} & B_{n-1} \end{array} \tag{7.2}$$

□

THEOREM 7.3.2. *Let $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C \to 0$ be a short exact sequences of chain complexes. Then there are natural maps $\partial_n : H_n(C) \to H_{n-1}(A)$ which makes the following sequence exact.*

$$\cdots \xrightarrow{g_n^*} H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{f_n^*} H_n(B) \xrightarrow{g_n^*} H_n(C) \xrightarrow{\partial_n} \cdots \tag{7.3}$$

Proof. Considering the exact sequences $\operatorname{coKer} d_n^A \to \operatorname{coKer} d_n^B \to \operatorname{coKer} d_n^C \to 0$ and $0 \to \operatorname{Ker} d_n^A \to \operatorname{Ker} d_n^B \to \operatorname{Ker} d_n^C$, and using Lemma 6.7.1 and 7.3.1, we get the desired result. □

7.4 *Splitting Chain Complex*

DEFINITION 7.4.1. Let $C_\bullet$ be a chain complex. We say $C_\bullet$ **splits** if there are morphisms $s_n : C_n \to C_{n+1}$ such that $d_n = d_n s_{n-1} d_n$.

PROPOSITION 7.4.2. *A chain complex $C_\bullet$ splits if and only if the first two short exact sequences in the Lemma 7.3.1 splits.*

Proof. Suppose that $C_\bullet$ splits, so there are morphisms $s_n : C_n \to C_{n+1}$ with $d_n = d_n s_{n-1} d_n$. Choose the natural morphisms $C_{n+1} \to B_n(C) \to Z_n(C) \to C_n$, and by composing s_n, choose the splitting maps for each morphisms. Then $C_{n+1} \to B_n \to Z_n \to C_n \xrightarrow{s_n} C_{n+1} \to B_n \to Z_n \to C_n = C_{n+1} \to B_n \to Z_n \to C_n$. Since $C_{n+1} \to B_n$ is epi and $B_n \to Z_n \to C_n$ are monic, $B_n \to Z_n \to C_n \xrightarrow{s_n} C_{n+1} \to B_n = 1_{B_n}$. Hence the sequences split.

Conversely, suppose that the sequences split. Choose $C_n \to Z_n \to B_n \to C_{n+1}$ as the splitting morphisms, and let the compositions as

s_n. Then, $d_n s_{n-1} d_n = C_n \to B_{n-1} \to Z_{n-1} \to C_{n-1} \to Z_{n-1} \to B_{n-1} \to C_n \to B_{n-1} \to Z_{n-1} \to C_{n-1}$. Deleting all the identities only give $C_n \to B_{n-1} \to C_n \to B_{n-1} \to C_{n-1}$, and deleting an identity again gives $C_n \to C_{n-1}$, which is the differential. □

7.5 Mapping Cones and Mapping Cylinders

Definition 7.5.1. Let $f_\bullet : C_\bullet \to D_\bullet$. The **mapping cone** of f is the chain complex $\mathrm{Cone}(f)_\bullet$, with $\mathrm{Cone}(f)_n := C_{n-1} \oplus D_n$ and $d_n^{\mathrm{Cone}} = (-d_{n-1}^C p_1, d_n^D p_2 - f_n p_1)$.

For the identity map $1_{C_\bullet}$, we write $\mathrm{Cone}(1_{C_\bullet})$ as $\mathrm{Cone}(C)$.

Theorem 7.5.2. *Let $f_\bullet : C_\bullet \to D_\bullet$ is a chain map. Then there is a following short exact sequence of chain complexes.*

$$0 \to D_\bullet \to \mathrm{Cone}(f) \to C[-1]_\bullet \to 0 \tag{7.4}$$

Therefore there is a following homology long exact sequence.

$$\cdots \to H_{n+1}(\mathrm{Cone}(f)) \to H_n(C) \to H_n(D) \to H_n(\mathrm{Cone}(f)) \to \cdots \tag{7.5}$$

Here, $H_n(C) \to H_n(D)$ is the map $H_n(f)$.

Proof. □

Proposition 7.5.3. *A chain map $f_\bullet : C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is exact.*

Proof. By the Theorem 7.5.2, $f_\bullet$ is a quasi-isomorphism if and only if, for all n, $H_n(C) \xrightarrow{\sim} H_n(D) \to H_n(\mathrm{Cone}(f)) \to H_{n-1}(C) \xrightarrow{\sim} H_n(D)$ is exact. This is equivalent to $\mathrm{Ker}(H_n(D) \to H_n(\mathrm{Cone}(f)) = H_n(D)$ and $\mathrm{Im}(H_n(\mathrm{Cone}(f) \to H_{n-1}(C)) = 0$, thus if and only if $H_n(\mathrm{Cone}(f)) = 0$, for all n. □

Proposition 7.5.4. $\mathrm{Cone}(C)$ *is split exact.*

Proof. Consider $s_n : \mathrm{Cone}(C)_n \to \mathrm{Cone}(C)_{n+1}$ defined as $(-p_2, 0)$. Then $d_n s_{n-1} d_n = (-d_{n-1}^C p_1, d_n^C p_2 - 1_{C_n} p_1)(-p_2, 0)(-d_{n-1}^C p_1, d_n^C p_2 - 1_{C_n} p_1) = (d_n^D p_2) = (-d_{n-1}^C p_1, d_n^C p_2 - 1_{C_n} p_1)(1_{C_n} p_1 - d_n^C p_2, 0) = (-d_{n-1}^C p_1, d_n^C p_2 - 1_{C_n} p_1)$, showing that Cone(C) splits. Furthermore, because $1_{C_\bullet}$ is a quasi-isomorphism, $\mathrm{Cone}(C)$ is exact by Proposition 7.5.3. □

7.6 Chain Homotopy

Definition 7.6.1. A chain map $f_\bullet : C_\bullet \to D_\bullet$ is **null homotopic** if there are morphisms $s_n : C_n \to D_{n+1}$ such that $f_n = d^D_{n+1} s_n + s_{n-1} d^C_n$.

For two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, if $(f - g)_\bullet$ is null homotopic, then we say $f_\bullet, g_\bullet$ are **chain homotopic**.

For a chain map $f_\bullet : C_\bullet \to D_\bullet$, if there is a chain map $g_\bullet : D_\bullet \to C_\bullet$ where $fg_\bullet$ and $gf_\bullet$ are chain homotopic to the identity maps, then we call $f_\bullet$ a **chain homotopy equivalence**.

Proposition 7.6.2. *A chain complex $C_\bullet$ is split exact if and only if $1_{C_\bullet}$ is null homotopic.*

Proof. Suppose that $C_\bullet$ splits. Then we have a collection of morphisms $s_n : C_n \to C_{n+1}$ satisfying $d_n = d_n s_{n-1} d_n$. □

Lemma 7.6.3. *A chain map $f_\bullet : C_\bullet \to D_\bullet$ is null homotopic if and only if there exists a map $(-s, f) : \mathrm{Cone}(C) \to D$.*

Theorem 7.6.4. *Let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map. Then $f_\bullet$ is null homotopic if and only if $H_n(f) : H_n(C) \to H_n(D)$ are zero. Therefore, $f_\bullet, g_\bullet$ are chain homotopic if and only if $H_n(f) = H_n(g)$.*

Proof. Suppose that $f_\bullet$ is a null homotopic chain map. Due to the Theorem 7.5.2, □

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Chapter 8

Group Homology and Cohomology

8.1 Definitions

Definition 8.1.1. Let G be a group. A G**-module** is an abelian group A on which G acts by additive maps on the left.

The category Mod_G is a category whose objects are G-module and morphisms are G-set maps.

A **trivial** G**-module** is a G-module A with $ga = a$ for all $g \in G, a \in A$.

A **trivial** G**-module functor** is a functor $T : \mathsf{Mod}_{\mathbb{Z}} \to \mathsf{Mod}_G$, taking an abelian group to a trivial G-module.

Definition 8.1.2. Let A be a G-module.

1. The **invariant subgroup** is a subgroup of A defined as following.

$$A^G := \{a \in A : ga = a, \forall (g, a) \in G \times A\} \tag{8.1}$$

2. The **coinvariants** is an abelian group defined as following.

$$A_G := A/G(\{(ga - a) : (g, a) \in G \times A\}) \tag{8.2}$$

Proposition 8.1.3.

1. *The map* $-^G : \mathsf{Mod}_G \to \mathsf{Mod}_{\mathbb{Z}}$ *is a functor.*
2. *The map* $-_G : \mathsf{Mod}_G \to \mathsf{Mod}_{\mathbb{Z}}$ *is a functor.*

Proof.

1. Let $f : A \to B$ be a G-set map. To show that the map $f^G : A^G \to B^G$ is naturally induced, we need to show that $ga = a$ implies $gf(a) = f(a)$. Because f is a G-set map, $gf(a) = f(ga) = f(a)$.
2. Let $f : A \to B$ be a G-set map. To show that the map $f_G : A_G \to B_G$ is naturally induced, we need to show that $ga - a \in A$ becomes $g'b - b = f(ga - a) \in B$ for some $g' \in G$ and $a \in A$. Because f is a G-set map, $f(ga - a) = gf(a) - f(a)$, thus $g' = g$ and $b = f(a)$ gives the desired result.

□

Theorem 8.1.4.

1. *The functor $-^G$ is right adjoint to the trivial module funtor, thus a left exact functor.*
2. *The functor $-_G$ is left adjoint to the trivial module functor, thus a right exact functor.*

Proof.

1. What we need to show is that $\mathsf{Mod}_{\mathbb{Z}}(A^G, B) \simeq \mathsf{Mod}_G(A, T(B))$ for any G-module A and $\mathbb{Z}$-module B. Take $f : A^G \to B$. The extension of f to $A \to T(B)$ exists, by taking $f(A \setminus A^G) = 0$. Suppose that there is another map $h : A \to T(B)$ such that the restriction $h|_{A^G} \to B$ is a zero map. Suppose that $h(a) \neq 0$ for some $a \in A$. By assumption, $ga - a \neq 0$, thus $h(ga) \neq h(a)$. But due to the triviality, $h(ga) = gh(a) = h(a)$, contradiction.
2. What we need to show is that $\mathsf{Mod}_G(T(A), B) \simeq \mathsf{Mod}_{\mathbb{Z}}(A, B_G)$ for any $\mathbb{Z}$-module A and G-module B. Take $f : T(A) \to B$. This map naturally extends to $A \to B_G$, because $gf(a) - f(a) = f(ga) - f(a) = f(a - a) = 0$, and this kind of extension is unique.

□

Lemma 8.1.5. *Let A be a G-module and $\mathbb{Z}$ be a trivial G-module. Then $A_G \simeq \mathbb{Z} \otimes_G A$ and $A^G \simeq G(\mathbb{Z}, A)$.*

Proof. By considering $\mathbb{Z}$ as a $\mathbb{Z} - G$ bimodule, the trivial G-module functor T can be written as $\mathbb{Z}(\mathbb{Z}, -)$ whose left adjoint is $\mathbb{Z} \otimes_G -$, as we can see on Proposition 5.3.1. Also, $A^G \simeq \mathbb{Z}(\mathbb{Z}, A^G) \simeq G(\mathbb{Z}, A)$ by adjointness in Theorem 8.1.4. □

Definition 8.1.6. Let A be a G-module. Then we write

$$H_*(G; A) := L_*(-_G)(A) \simeq \mathrm{Tor}_*^G(\mathbb{Z}, A) \tag{8.3}$$

and call them the **homology groups of G with coefficients in A**. Similarly, we write

$$H^*(G; A) := R^*(-^G)(A) \simeq \mathrm{Ext}_G^*(\mathbb{Z}, A) \tag{8.4}$$

and call them the **cohomology groups of G with coefficients in A**.

Part III

Categorical Homology

Chapter 9

The Derived Category

9.1
Triangulated Categories

Definition 9.1.1. A **category with translation** (C, T) is a category C with an equivalence of categories $T : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$, called the **translation functor**.

For two categories with translations, a **functor** $F : (\mathsf{C}, T) \to (\mathsf{D}, S)$ of translation categories is a functor $F : \mathsf{C} \to \mathsf{D}$ satisfying $FT = SF$.

For two functors $F, G : (\mathsf{C}, T) \to (\mathsf{D}, S)$, a **natural transformation** $\epsilon : F \to F'$ of translation functors is a natural transformation which makes $FT \xrightarrow{\epsilon T} GT \xrightarrow{\sim} SG$ and $FT \xrightarrow{\sim} SF \xrightarrow{S\epsilon} SG$ same.[1]

1

$$\begin{array}{ccc} FT & \xrightarrow{\epsilon T} & GT \\ \downarrow\sim & & \downarrow\sim \\ GF & \xrightarrow{S\epsilon} & SG \end{array}$$

Definition 9.1.2. Let (C, T) be an additive category with translation. A **triangle** in D is a sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$, and their morphism is a collection of maps $X \xrightarrow{\alpha} X'$, $\beta : Y \xrightarrow{\beta} Y'$, and $Z \xrightarrow{\gamma} Z'$, satisfying $\beta f = f'\alpha$, $\gamma g = g'\beta$, $T(\alpha)h = h'\gamma$.[2]

2

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \downarrow T(\alpha) \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' \end{array}$$

Definition 9.1.3. A **triangulated category** is an additive category (C, T) with a family of triangles, called **distinguished triangles**, satisfying the followings.

1. A triangle isomorphic to a distinguished triangle is a distinguished triangle.
2. $X \xrightarrow{1_X} X \to 0 \to TX$ is a distinguished triangle.
3. For all $f : X \to Y$, there is a distinguished triangle $X \xrightarrow{f} Y \to Z \to TX$.
4. A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a distinguished triangle if and only if $Y \xrightarrow{-g} Z \xrightarrow{-h} TX \xrightarrow{-T(f)} TY$ is a distinguished triangle.
5. For two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y \xrightarrow{g'} Z \xrightarrow{h'} TX$, and morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ satisfying $f'\alpha = \beta f$, there is a morphism $\gamma : Z \to Z'$ which gives a morphism between triangles.[3]

3

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow\alpha & & \downarrow\beta & & \vdots\gamma & & \downarrow T(\alpha) \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' \end{array}$$

6. For three distinghished triangles,

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \to TX$$
$$Y \xrightarrow{g} Z \xrightarrow{k} X' \to TY$$
$$X \xrightarrow{gf} Z \xrightarrow{l} Y' \to TX$$

there is a distinguished triangle

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} TZ' \tag{9.1}$$

making the diagram[4], where the triangles are rows and the third vertical triangle is the last given distinguished triangle, commute.[5]

A **triangulated functor** of triangulated categories is a functor of additive categories with translation, sending distinguished triangles to distinguished triangles.

Proposition 9.1.4. *$X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ is a distinguished triangle implies $gf = 0$.*

Proof. We have a distinguished triangle $X \xrightarrow{1_X} X \to 0 \to TX$, and a map from it to our given triangle, constructed with $X \xrightarrow{1_X} X$ and $X \to f$.[6] This shows $gf = 0$ directly. □

Definition 9.1.5. Let (C, T) be a triangulated category and D be an abelian category. Then an additive functor $F : \mathsf{C} \to \mathsf{D}$ is **cohomological** if for any distinguished triangles $X \to Y \to Z \to TX$ in C the sequence $F(X) \to F(Y) \to F(Z)$ is exact in D.

Proposition 9.1.6. *For any $C \in \mathsf{C}$, $\mathsf{C}(C, -)$ and $\mathsf{C}(-, C)$ are cohomological.*

Proof. Let $X \to Y \to Z \to TX$ be a distinguished triangle. To show that

$$\mathsf{C}(C, X) \xrightarrow{f_*} \mathsf{C}(C, Y) \xrightarrow{g_*} \mathsf{C}(C, Z) \tag{9.2}$$

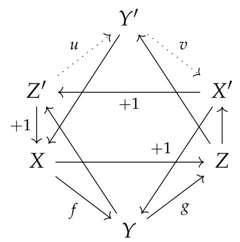
is exact, we need to show that for all $\varphi \in \mathsf{C}(C, Y)$ with $g\varphi = 0$, there is $\psi : C \to X$ such that $\varphi = f\psi$. But from the two sequences $C \xrightarrow{1_C} C \to 0 \to TC$ and $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$,by the conditions of distinguished triangle, there is a map $\psi : C \to X$ which is $\varphi = f\psi$[7]. □

Proposition 9.1.7. *For a cohomological functor F and a distinguished triangle $X \to Y \to Z \to TX$, there is a long exact sequence*

$$\cdots \to F(T^{-1}Z) \to F(X) \to F(Y) \to F(Z) \to F(TX) \to \cdots . \tag{9.3}$$

[4]
$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \longrightarrow & TX \\
\downarrow 1_X & & \downarrow g & & \downarrow u & & \downarrow 1_{TX} \\
X & \xrightarrow{gf} & Z & \xrightarrow{l} & Y' & \longrightarrow & TX \\
\downarrow f & & \downarrow 1_Z & & \downarrow v & & \downarrow T(f) \\
Y & \xrightarrow{g} & Z & \xrightarrow{k} & X' & \longrightarrow & TY \\
\downarrow h & & \downarrow l & & \downarrow 1_{X'} & & \downarrow T(h) \\
Z' & \xrightarrow{u} & Y' & \xrightarrow{v} & X' & \xrightarrow{w} & TZ'
\end{array}$$

[5] We may write this diagram in the form of octahedron. In the following diagram, $X' \xrightarrow{+1} Y$ is a morphism $X' \to TY$.



[6]
$$\begin{array}{ccccccc}
X & \xrightarrow{1_X} & X & \longrightarrow & 0 & \longrightarrow & TX \\
\downarrow 1_X & & \downarrow f & & \downarrow & & \downarrow 1_{TX} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX
\end{array}$$

[7]
$$\begin{array}{ccccccc}
C & \xrightarrow{1_C} & C & \longrightarrow & 0 & \longrightarrow & TC \\
\downarrow \psi & & \downarrow \varphi & & \downarrow & & \downarrow T\psi \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX
\end{array}$$

Proof. This directly follows from the definition of cohomological functor and distinguished triangle. □

9.2
Complexes and Mapping cone

| DEFINITION 9.2.1. | Let (C, T) be an additive category with translation.

1. A **differential object** is an object $C \in \mathsf{C}$ with a morphism $d_C : C \to TC$.
2. A **morphism of differentials** is a differential morphism $f : C \to D$ between complexes.
3. A differential object C is a **complex** if $T(d_C)d_C = 0$.
4. A **morphism of complexes** is a morphism $f : C \to D$ such that $T(f)d_C = d_D f$.[8]

8

$$\begin{array}{ccc} C & \xrightarrow{d_C} & TC \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle T(f)} \\ D & \xrightarrow{d_D} & TD \end{array}$$

| DEFINITION 9.2.2. | Let (C, T) be an additive category with translation. For a differential object C, the differential object TC with the differential $d_{TC} := -T(d_C)$ is called the **shifted object** of C.

| DEFINITION 9.2.3. | Let (C, T) be an additive category with translation, and there are two differential objects C, D with a morphism $f : C \to D$. Then the **mapping cone** $\mathrm{Cone}(f)$ is the object $TC \oplus D$ with differential

$$d_{\mathrm{Cone}(f)} := \begin{bmatrix} d_{TC} & 0 \\ T(f) & d_D \end{bmatrix}. \tag{9.4}$$

Define $\alpha(f) : D \to \mathrm{Cone}(f)$ as $\alpha(f) := 0 \oplus 1_D$ and $\beta(f) : \mathrm{Cone}(f) \to TC$ as $\beta(f) = (1_{TX}, 0)$. Then a triangle

$$C \xrightarrow{f} D \xrightarrow{\alpha(f)} \mathrm{Cone}(f) \xrightarrow{\beta(f)} TC \tag{9.5}$$

exists, and we call it a **mapping cone triangle**.

| PROPOSITION 9.2.4. | *Let (C, T) be an additive category with translation. For a complexes C, D with $f : C \to D$,* $\mathrm{Cone}(f)$ *is a complex if and only if f is a morphism of complexes.*

Proof. Because

$$T(d_{\mathrm{Cone}(f)})d_{\mathrm{Cone}(f)} = \begin{bmatrix} T(d_{TC}) & 0 \\ T^2(f) & T(d_D) \end{bmatrix} \begin{bmatrix} d_{TC} & 0 \\ T(f) & d_D \end{bmatrix}, \tag{9.6}$$

$\mathrm{Cone}(f)$ is a complex if and only if

$$T(-T(f)d_C + d_D f) = 0 \tag{9.7}$$

which is equivalent with $T(f)d_C = d_D f$, that is, f is a morphism of complexes. □

9.3
The Homotopy Category

Lemma 9.3.1. *Let (C, T) be an additive category with translation, with differential objects C, D and a morphism $u : C \to T^{-1}D$. Define*

$$f := T(u)d_C + T^{-1}(d_D)u. \tag{9.8}$$

Then f is a differential morphism if and only if

$$d_D T^{-1}(d_D)u = T^2(u)T(d_C)d_C. \tag{9.9}$$

Thus, if C and D are complexes, then f is a morphism of complexes.

Proof. This directly follows from the definition. □

Definition 9.3.2. Let (C, T) be an additive category with translation, with two differential objects C, D. Then a differnetial morphism $f : C \to D$ is **zero homotopic** if there is a morphism $u : C \to T^{-1}D$ with satisfying

$$f = T(u)d_C + T^{-1}(d_D)u. \tag{9.10}$$

We say two differential morphisms $f, g : C \to D$ are **homotopic equivalent** or **homotopic** if $f - g$ is zero homotopic.

Proposition 9.3.3. *Let $f : C \to D$ and $g : D \to E$ be differential objects. If f or g is zero homotopic, then gf is zero homotopic.*

Proof. Let $f = T(u)d_C + T^{-1}(d_D)u$ with $u : C \to T^{-1}D$. Then,

$$\begin{aligned} gf &= gT(u)d_C + gT^{-1}(d_D)u \\ &= gT^{-1}(u)d_C + T^{-1}(d_E)T^{-1}(g)u \\ &= T(T^{-1}(g)u)d_C + T^{-1}(d_E)(T^{-1}(g)u). \end{aligned}$$

This shows the desired result. The g zero homotopic case is similar. □

Definition 9.3.4. Let (C, T) be an additive category with translation. Then the **homotopy category** $K_d(\mathsf{C})$ is a category with objects as differential objects, and morphisms as differential morphisms quotiented by homotopy equivalence.

Proposition 9.3.5. *Let (C, T) be an additive category with translation. Then $(K_d(\mathsf{C}), T)$ is also an additive category with translation.*

Proof. The quotient of abelian group is an abelian group, hence the morphism set is an abelian group. Also, the translation functor on C naturally induces the translation functor on $K_d(\mathsf{C})$. □

Theorem 9.3.6. *Defining a set of distinguished triangles of* $(K_d(\mathsf{C}), T)$ *as the set of triangles isomorphic to a mapping cone triangle gives a triangulated category.*

Proof. □

Definition 9.3.7. Let $K_d(\mathsf{C})$ be a homotopy category. Then the **chain homotopy category** $K_c(\mathsf{C})$ is a triangulated full subcategory of $K_d(\mathsf{C})$ consisting of complexes in (C, T), with induced family of distinguished triangles.[9]

[9] $K_c(\mathsf{C})$ is triangulated because the mapping cone of a complex morphism is a complex.

Proposition 9.3.8. *Let* $F : (\mathsf{C}, T) \to (\mathsf{D}, S)$ *be a functor of additive categories with translation. Then* F *defines naturally triangulated functors* $K_d(F) : K_d(\mathsf{C}) \to K_d(\mathsf{D})$ *and* $K_c(F) : K_c(\mathsf{C}) \to K_c(\mathsf{D})$.

Proof. Because F sends a zero homotopic morphism to a zero homotopic morphism, we only need to show that F sends a mapping cone triangle to mapping cone triangle, which follows from the definition of mapping cone triangle. □

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